January 1970

Solution to Transient Vertical Moisture Movement Based Upon Saturation–Capillary Pressure Data and Modified Burdine Theory

Roland W. Jeppson

SOLUTION TO TRANSIENT VERTICAL MOISTURE MOVEMENT BASED UPON SATURATION-CAPILLARY PRESSURE DATA AND A MODIFIED BURDINE THEORY

By

Roland W. Jeppson

The work reported herein was carried out under cooperative agreement No. 12-14-100-9715(41) between the Utah Water Research Laboratory and the Agricultural Research Service with funds from the Soil and Water Conservation Research Division, Northwest Watershed Research Center, Boise, Idaho, 83701.

Utah Water Research Laboratory
College of Engineering
Utah State University
Logan, Utah 84321

December 1970

PRWG-59c-5

Abstract

Using finite differences and the Crank-Nicolson implicit scheme for solving parabolic type partial differential equations, a computer program has been developed for solving the one-dimensional, vertical movement of water in soils. The formulation of the initial boundary value problem is obtained by introducing a new dependent variable through the Kirchhoff transformation to replace the hydraulic head. Data relating saturation (or moisture content) to the capillary pressure in the soil are used to define the hydraulic properties of the soil which are needed in order to obtain a solution. The Burdine Theory has been implemented in the program to obtain the needed relationship of hydraulic conductivity to capillary pressure. This formulation and solution method is consistent with the solution method developed earlier for three dimensional axisymmetric movement of water applied at the surface by a circular infiltrometer, so that comparisons of the solution results from the two different cases would indicate quantitative effects on the flow pattern of the component of radial moisture movement.

A number of solutions have been obtained for several initial moisture contents and for several application rates (including variable rates equal to the intake capacity of the soil), for a soil at the Reynold's Creek experimental watershed. Saturation-capillary pressure data for this soil were obtained by the Agricultural Research Service in the laboratory. Laboratory measurements of the hydraulic conductivity corresponding to number of capillary pressures were also obtained. Using the saturation-capillary pressure data in the Burdine equations for evaluating the hydraulic conductivity gives good agreement with these latter laboratory measurements.

The results from these solutions have been used to display the variations of hydraulic head, saturation and hydraulic gradient with time under varying conditions. By contrasting the results from these solutions with those from similar solutions for the axisymmetric case, the effects on the flow patterns due to the radial component of moisture movement have been determined.

Table of Contents

Notation

A - Matrix

$\vec{B}$ - Vector

D - Depth of soil

h - Hydraulic head equal to elevation head plus pressure head i.e., $h = y + p$ (L)

$i = y/\Delta y + 1$ - Subscript denoting space increment

$j = \tau/\Delta\tau + 1$ - Superscript denoting time step

k - Subscript denoting space increment just beyond wetting front

K - Hydraulic conductivity (L/T)

K_o - Saturated hydraulic conductivity (L/T)

$K_r = \frac{K}{K_o}$ - Relative hydraulic conductivity

$n = D/\Delta y + 1$ - Subscript denoting space increment corresponding with impervious layer

p - Capillary pressure head; is a negative quantity for partially saturated flow and becomes positive for saturated flow (pressure/specific weight) (L)

p_1 - Reference pressure used in Kirchhoff transformation (F/L^2)

p_o - Reference pressure used as lower limit in Burdine integrals (F/L^2)

p_b - Bubbling pressure head (L)

q - Application rate; volumetric flow rate per unit area (L/T)

S - Saturation; volume of water/volume of voids

S_o - Reference saturation used as upper limit in Burdine integral

$S_e = (S-S_r) / (1-S_r)$ - Effective saturation

S_r - Residual saturation

t - Time (T)

y - Vertical cartesian coordinate (L)

α - Parameter

Δ - Difference operator

ζ - Parameter

η - Soil porosity; volume of void/total volume

λ - Pore size distribution exponent

$\vec{\xi}$ - Vector

ξ - Dependent variable introduced by Kirchhoff Transformation (L)

ξ_o - Initial distribution of ξ .

$\tau = K_o t$ - Time parameter (L)

$\nabla \equiv \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{j}$ - Vector operator

∂ - Partial differentiation operator

SOLUTION TO TRANSIENT VERTICAL MOISTURE MOVEMENT BASED UPON SATURATION-CAPILLARY PRESSURE DATA AND A MODIFIED BURDINE THEORY

Introduction

Recently under a cooperative agreement with the Agricultural Research Service, (Northwest Watershed Research Center), computer programs have been developed for solving problems of three-dimensional axisymmetric transient moisture movement from circular infiltrometers through partially saturated soils (Jeppson, 1970 a & b). These programs use finite difference methods in solving the nonlinear partial differential equations. The results from these solutions are being used in conjunction with field infiltrometer measurements by ARS - Northwest Watershed Research Center in determining the hydraulic properties of watershed soils in fundamental studies with the ultimate objective of developing a computer simulated model of a watershed flow system. In conjunction with this research program, it is necessary to know the quantitative effects of the radial component of flow from a circular infiltrometer in order to correlate the results with field situations in which the moisture is applied over a large area resulting in only vertical movement. Such influences will vary depending upon the hydraulic properties of the soil.

The magnitude of the various influences can be obtained by studying and contrasting the solution results for the case of moisture movement from a circular infiltrometer with the results from similar solutions based upon the assumption that the moisture movement is one-dimensional. While a number of solutions and computer programs are available for solving the one-dimensional problem, it appeared that less total effort would be required to develop a computer program, than to adapt an available computer program to give results consistent with the solution results for the axisymmetric problem. In part this decision was reached because the best overall solution results for the circular infiltrometer problem are obtained from a formulation based on the Kirchhoff transformation (this transformation is discussed later) when laboratory data for capillary pressure and saturation for a particular soil are used in a table look-up technique in conjunction with a modified Burdine Theory (also discussed later) in defining certain required relationships for the hydraulic properties of the soil. The writer is not aware of available solutions based on this approach, yet in order to readily contrast the two situations the solutions must be based on the same assumptions. Furthermore, with the methods available for solving one-dimensional problems a small amount of effort is required in the development of such a computer program.

The method of solution for the one-dimensional problem is discussed in this paper, along with the modifications made in the Burdine Theory for handling the imbibition case. A limited amount of analyses and presentation of results and comparisons are given in this paper. After more laboratory and field data become available from laboratory and experiments at the ARS Northwest Watershed Research Center, a subsequent report will present more extensive analyses and conclusions.

Formulation

The following differential equation for defining the vertical moisture movement in porous media is obtained by using Darcy's Law in conjunction with the one-dimensional continuity equation

$$\nabla \cdot (K_r \nabla h) = \frac{\partial}{\partial y} \left(K_r \frac{\partial h}{\partial y}\right) = \frac{\eta}{K_o} \frac{\partial S}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which h is the total hydraulic head and equals the sum of the pressure and elvation heads, i.e., $h = p + y$; η is the soil porosity, i.e., the volume of voids per unit volume, S is the degree of saturation and equals the volume of water in the soil divided by the volume of soil voids; K_o is the saturated hydraulic conductivity of the soil with units of velocity; and K_r is the relative hydraulic conductivity whose magnitude equals the hydraulic conductivity of the partially saturated soil divided by its saturated hydraulic conductivity. Studies (Brooks and Corey, 1964 and 1966; King, 1965; White, et al., 1970) have demonstrated that the relative hydraulic conductivity can be defined adequately for most applications as a function of the capillary pressure head, p.

The left side of Eq. 1 can be simplified by introduction of the dependent variable ξ in place of h by means of the Kirchhoff transformation. This transformation is accomplished by letting

$$\frac{d\xi}{dp} = K_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

or

$$\xi = \int_{p_1}^{p} K_r(p)\,dp \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

(in which p_1 is a reference pressure, generally equal zero, at which partial saturation first occurs) and then noting that

$$\nabla\xi = (d\xi/dp)\nabla p \quad \text{and} \quad \partial h/\partial y = \partial p/\partial y + 1$$

Equation 1 becomes

$$\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial K_r}{\partial y} = \frac{\eta}{K_o}\frac{\partial S}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since, however, K_r and S are functions of p, $\partial K_r/\partial y = (\partial K_r/\partial p)(\partial p/\partial \xi)(\partial \xi/\partial y) = (\partial K_r/\partial p)(1/K_r)(\partial \xi/\partial y)$ and $\partial S/\partial t = (\partial S/\partial p)(1/K_r)(\partial \xi/\partial t)$, Eq. 3 can be written as

$$\frac{\partial^2 \xi}{\partial y^2} + \frac{1}{K_r}\frac{\partial K_r}{\partial p}\frac{\partial \xi}{\partial y} = \frac{\eta}{K_r}\frac{\partial S}{\partial p}\frac{\partial \xi}{\partial \tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which the time parameter $\tau = K_o t$ for homogeneous soil.

The solution to Eq. 4 is dependent upon obtaining values for $\partial S/\partial p$, K_r and $\partial K_r/\partial p$. Of these needed quantities it is practical to measure capillary tension and saturation in the laboratory or field with instruments such as tensiometers, Neutron meters or gamma probes from which values of $\partial S/\partial p$ as related to ξ can be obtained. Hydraulic conductivity and capillary pressure data, however, are much more difficult to obtain. Consequently, it is very desirable to obtain values for K_r and $\partial K_r/\partial p$ as related to ξ from saturation pressure data, or from some functional relationship involving pressure. The functional relationships proposed in the literature do not give close agreement to the data obtained for soils in the Reynolds Creek Experimental Watershed operated by ARS-Northwest Watershed Research Center. Reasonably close agreement seems to result, however, from a modification of the Burdine Theory (Burdine, 1952) from capillary pressure-saturation data. The approximations involved in the Burdine Theory and its application are discussed by Brooks and Corey (1964 and 1966) and Laliberte, et al. (1968). The results from the Burdine Theory are given by the following equation:

$$K_r = S_e^{\,2} \frac{\int_{S_r}^{S} \frac{dS}{p^2}}{\int_{S_r}^{1} \frac{dS}{p^2}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

in which S_e is the effective saturation defined by $S_e = (S - S_r)/(1 - S_r)$, in which S_r is the residual saturation. The value of S_r is generally determined to give as good a fit as possible to a functional relationship between S_e and p, but physically it is the saturation at which moisture movement ceases.

The applications of the Burdine Theory in the references cited, above have been for the case of desaturation. For problems dealing with imbibition, as is the case for the infiltration problem of this paper, it became clear that Eq. 5 must be modified, since the capillary pressure head p becomes zero for values of S generally slightly less than unity and this would result in a division by zero. Two modifications have been introduced to Eq. 5. First a constant pressure head p_o has been added to the pressure head and second, the upper limit of the integral in the denominator of Eq. 5 has been changed to S_o a value at which the pressure head, p, becomes zero and whose magnitude is generally slightly less than unity. With these modifications the Burdine Integrals become,

$$K_r = S_e^2 \frac{\int_{S_r}^{S} \frac{dS}{(p+p_o)^2}}{\int_{S_r}^{S_o} \frac{dS}{(p+p_o)^2}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Introduction of S_o and p_o in the Burdine Integrals results in additional parameters which are needed in the description of the hydraulic properties of a soil. A value for S_o can be obtained from saturation pressure data when $p = 0$. Studies are needed to relate the value of p_o to the physical properties or to some measurable hydraulic characteristic of the soil. In the absence of such studies, the value of p_o must be based on judgment guided by values obtained by trial for soil for which hydraulic conductivity-capillary pressure data are also available. From analysis of a limited amount of data it appears that values for the relative hydraulic conductivity are not highly sensitive to small changes in the value of p_o particularly in the region in which K_r is not too much less than unity. Furthermore, since little flux movement exists in regions in which K_r is very small and because of the lack of sensitivity in regions in which K_r approaches unity, it is believed that a reasonable estimate of p_o will be adequate for many applications of Eq. 6 and, consequently, hydraulic conductivity data will not be necessary.

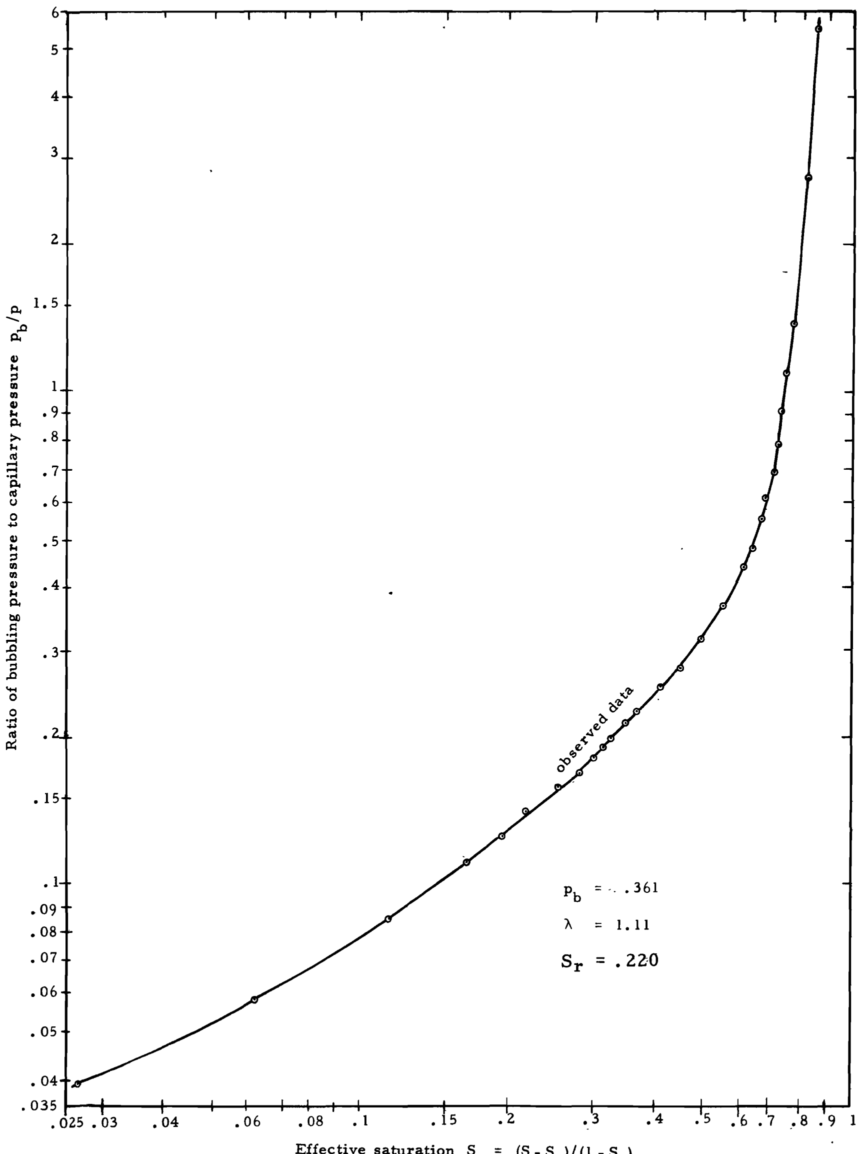


Fig. 1 Capillary pressure - effective saturation relationship of experimental data from soil at Reynold's Creek experimental watershed

Saturation pressure data (see Table 1 and Fig. 1) from soil at the ARS Reynolds Creek experimental watershed were used in a numerical evaluation of Eq. 6 using several values for p_o and the results compared with laboratory data for hydraulic conductivity of the same soil. In the numerical evaluation of Eq. 6 a second degree polynomial is passed through each three consecutive points and this polynomial integrated as described by Jeppson (1970c). The integration over the interval between S_r and the smallest value of S contained in the saturation data is carried out by means of the Brooks-Corey Equation (Brooks and Corey, 1966). Table 2 gives the results from Eq. 6 for $p_o = 1$ and $S_o = 0.939$. Values for $(1/K_r)(\partial K_r/\partial p)$ and $(1/K_r)(\partial S/\partial p)$ which are given in Table 2 are required to obtain a solution to Eq. 4.

The results from this comparison are shown in Fig. 2. Increasing values of p_o increases values of K_r, as produced by Eq. 6, over the entire range of capillary pressure heads but more markedly for the large negative values of pressure head, whereas decreases in the value of S_o used in Eq. 6 increase K_r more markedly in that portion of the curve where p approaches zero. Relatively good agreement exists between the values of K_r determined by Eq. 6 and the experimental data. Therefore, Eq. 6 has been implemented in solving the infiltration of moisture through soils in a table look-up technique. The techniques used in this table look-up procedure are identical to those for solving the axisymmetric problem (Jeppson, 1970 a & b).

The mathematical problem of one-dimensional vertical moisture movement through soils is depicted in Fig. 3 in the τy plane. The differential equation to be solved is shown in the rectangle within the τy plane and the initial and boundary conditions are given by an equation adjacent to the applicable boundary of the region of the problem.

The initial condition, namely $\xi = \xi_o(y)$, represents the distribution of ξ as a function of y which exists prior to the application of moisture to the surface. If the soil moisture is in equilibrium under the action of only the gravitational force, the hydraulic head will be constant throughout the soil. This constant hydraulic head condition has been used in

Table 1. Capillary pressure-saturation data for soil at the Reynolds Creek experimental watershed. Data were obtained by Niel Biggs from laboratory tests of disturbed samples. The original data has been smoothed to provide for continuous polynomial curve fitting between each adjacent three data values.

Capillary Pressure Head (feet)	0	.0656	.131	.194	.259
Degree of Saturation (percent)	.939	.882	.852	.832	.817

Capillary Pressure Head (feet)	.325	.390	.456	.522	.587	.650	.748	.813	.978	1.138
Degree of Saturation (percent)	.796	.788	.775	.762	.750	.732	.711	.692	.649	.608

Capillary Pressure Head (feet)	1.302	1.434	1.594	1.693	1.759	1.824	1.890	1.988	2.152	2.280
Degree of Saturation (percent)	.572	.540	.508	.492	.485	.475	.466	.455	.439	.418

Capillary Pressure Head (feet)	2.543	2.887	3.258	4.206	6.114	9.288
Degree of Saturation (percent)	.391	.374	.349	.310	.268	.241

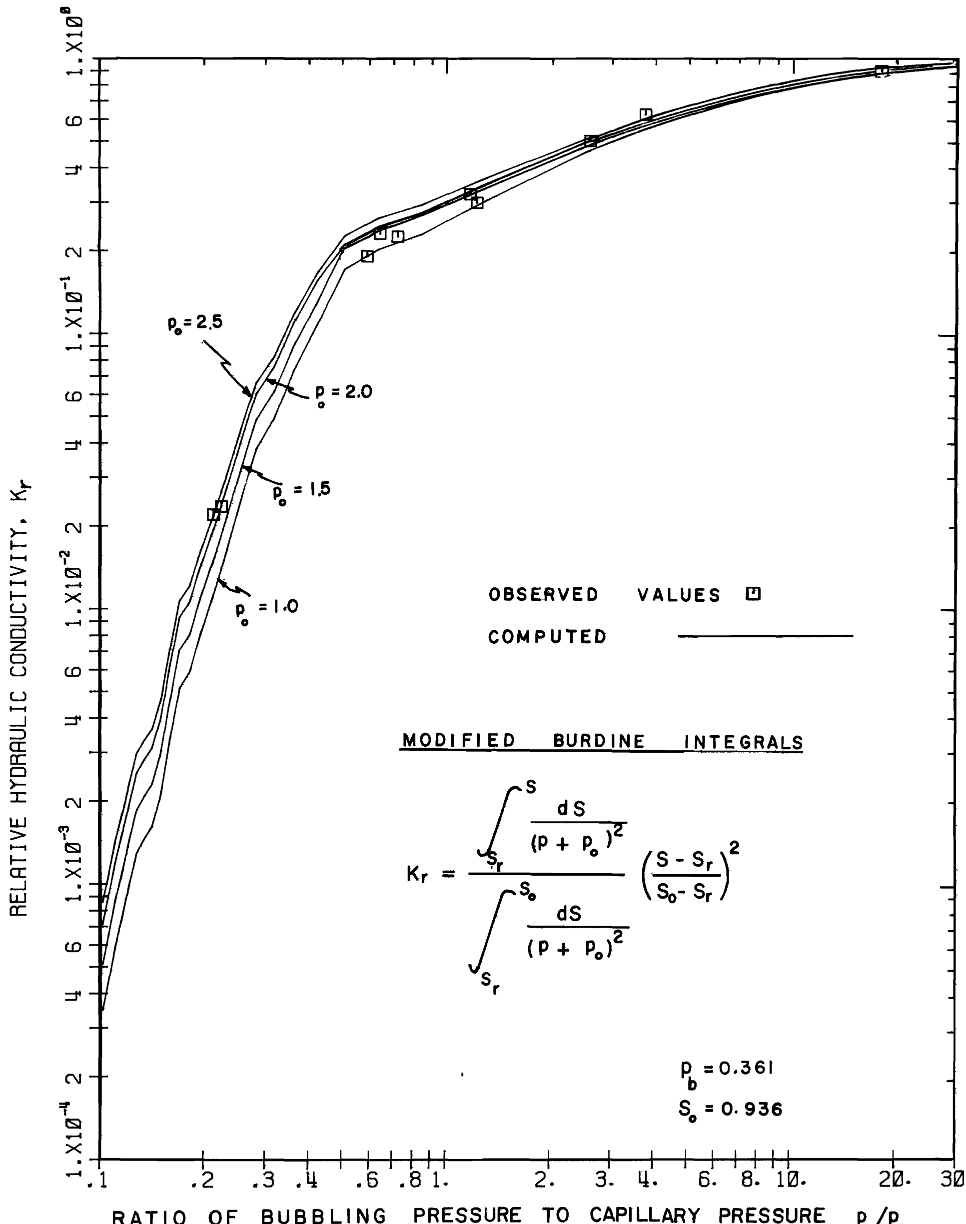


FIG. 2. COMPARISON OF HYDRAULIC CONDUCTIVITY COMPUTED FROM PRESSURE-SATURATION DATA BY THE BURDINE THEORY WITH OBSERVED VALUES.

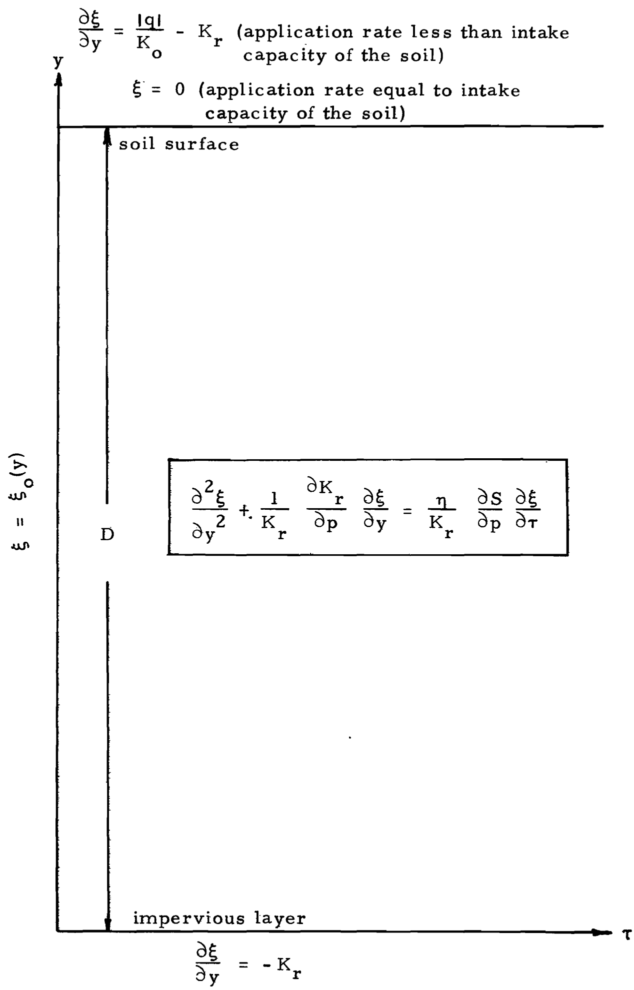


Fig. 3. Formulation of the initial boundary value problem for vertical movement of moisture in partially saturated soil.

the computer program for initiating the problem if another initialization is not supplied as input data. Initial values for $\xi = \xi_o(y)$ are obtained from the constant head condition through Eq. 2b by numerically evaluating the integral. In implementation of the table look-up technique values for ξ (as well as K_r, $(1/K_r)(\partial K_r/\partial p)$ and $(1/K_r)(\partial S/\partial p)$) are stored in the core memory of the computer for a specified number of increments of the pressure head. Over the first portion of the entire pressure head range of the problem (that is the portion in which the pressure approaches zero) the increment of pressure between adjacently stored values is one-half as great as over the latter portion of the range. Values for $\xi = \xi_o(y)$ are obtained by first computing the pressure head corresponding to y and subsequently, upon determining the indexes of the table whose adjacent pressure head entries bracket the computed pressure head, determining the value $\xi = \xi_o(y)$ by linear interpolation.

In applying a constant head initial condition one should be aware that watershed soils would approach this condition only toward the end of relatively long periods of no precipitation during which evapotranspiration has essentially also ceased. During and shortly after precipitation, hydraulic gradients will be positive resulting in larger values of the hydraulic head near the surface, whereas when evapotranspiration is occurring the hydraulic head will increase with depth below the surface. An extensive program of field measurements of soil moisture and soil tension by Renger, et al. (1970) clearly demonstrated the quantitative dependency of the distribution of hydraulic head within the soil profile with precipitation patterns as well as variations of soil types at different depths. Should such an initial distribution be known by measurements, or other means, it should be used as the initial condition. All results given in this paper, however, have been obtained assuming h is constant throughout the soil profile.

The boundary condition at the bottom of the soil profile at the impervious layer is obtained by noting that the vertical velocity of moisture movement at this boundary equals zero. From Darcy's Law, therefore,

Table 2. Values of the quantities stored in the core memory of the computer in implementing the table look-up technique for a soil with the saturation pressure data given in Table 1.

No.	Pressure head	Saturation	ξ	$\frac{1}{K_r}\frac{\partial S}{\partial P}$	$\frac{1}{K_r}\frac{\partial K_r}{\partial P}$	K_r
1	.0000	.9390	.00000	.21329+01	.16850+02	.99297+00
2	.1407	.8482	.10205+00	.81939+00	.36632+01	.45736+00
3	.2815	.8041	.15254+00	.84036+00	.25956+01	.29902+00
4	.4222	.7817	.18891+00	.86591+00	.20479+01	.22879+00
5	.5629	.7549	.21895+00	.93286+00	.17889+01	.20153+00
6	.7036	.7212	.24534+00	.12573+01	.19364+01	.17039+00
7	.8444	.6839	.26493+00	.24531+01	.26825+01	.10781+00
8	.9851	.6469	.27749+00	.37481+01	.30185+01	.73772-01
9	1.1258	.6110	.28598+00	.48572+01	.28480+01	.48899-01
10	1.2665	.5802	.29204+00	.59364+01	.27449+01	.38063-01
11	1.4073	.5459	.29643+00	.92701+01	.30860+01	.24561-01
12	1.5480	.5163	.29924+00	.11539+02	.28047+01	.16094-01
13	1.6887	.4925	.30112+00	.10975+02	.20103+01	.11047-01
14	1.8295	.4742	.30245+00	.18477+02	.26599+01	.81013-02
15	1.9702	.4569	.30342+00	.18295+02	.20652+01	.58811-02
16	2.1109	.4436	.30419+00	.20530+02	.19968+01	.51333-02
17	2.2516	.4219	.30479+00	.42447+02	.30222+01	.33200-02
18	2.3924	.4021	.30515+00	.43613+02	.22596+01	.21096-02
19	2.5331	.3916	.30542+00	.38908+02	.16449+01	.16158-02
20	2.6738	.3849	.30563+00	.33066+02	.12427+01	.14472-02
21	2.8146	.3779	.30583+00	.40641+02	.13562+01	.12899-02
22	2.9553	.3683	.30599+00	.77908+02	.21412+01	.10018-02
23	3.0960	.3584	.30611+00	.83962+02	.18715+01	.75696-03
24	3.2367	.3502	.30620+00	.93417+02	.17209+01	.58841-03
25	3.3775	.3416	.30628+00	.12454+03	.18574+01	.44548-03
26	3.5182	.3345	.30633+00	.13218+03	.16480+01	.34839-03
27	3.6589	.3284	.30638+00	.14415+03	.15180+01	.27901-03
28	3.7996	.3231	.30641+00	.15943+03	.14325+01	.22679-03
29	3.9404	.3182	.30644+00	.17788+03	.13732+01	.18621-03
30	4.0811	.3137	.30646+00	.19966+03	.13312+01	.15397-03
31	4.2218	.3068	.30648+00	.10738+04	.57335+01	.11260-03
32	4.3626	.2985	.30650+00	.52815+03	.21181+01	.73864-04
33	4.5033	.2938	.30651+00	.49769+03	.16579+01	.56928-04
34	4.7847	.2872	.30653+00	.53868+03	.13269+01	.37712-04
35	5.0662	.2820	.30654+00	.62833+03	.11903+01	.26524-04
36	5.3477	.2777	.30654+00	.75426+03	.11205+01	.19180-04
37	5.6291	.2738	.30655+00	.92089+03	.10848+01	.14072-04
38	5.9106	.2703	.30655+00	.11392+04	.10712+01	.10394-04
39	6.1920	.2671	.30655+00	.14270+04	.10746+01	.76874-05
40	6.4735	.2641	.30656+00	.18113+04	.10933+01	.56682-05
41	6.7549	.2613	.30656+00	.23343+04	.11278+01	.41483-05
42	7.0364	.2587	.30656+00	.30640+04	.11808+01	.29991-05
43	7.3178	.2562	.30656+00	.41159+04	.12579+01	.21293-05
44	7.5993	.2537	.30656+00	.56994+04	.13698+01	.14726-05
45	7.8808	.2514	.30656+00	.82321+04	.15381+01	.97971-06
46	8.1622	.2492	.30656+00	.12668+05	.18115+01	.61359-06
47	8.4437	.2471	.30656+00	.21721+05	.23270+01	.34577-06
48	8.7251	.2450	.30656+00	.47152+05	.36730+01	.15424-06
49	9.0066	.2430	.30656+00	.32356+06	.17543+02	.21808-07
50	9.2880	.2410	.30656+00	.48249+06	.20634+02	.14214-07

$$\frac{\partial h}{\partial y} = \frac{\partial p}{\partial y} + 1 = \frac{\partial p}{\partial \xi}\frac{\partial \xi}{\partial y} + 1 = \frac{1}{K_r}\frac{\partial \xi}{\partial y} + 1 = 0 \quad . \quad . \qquad (7a)$$

or

$$\frac{\partial \xi}{\partial y} = - K_r \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (7b)$$

If no impervious layer exists in a particular problem the depth D should be specified larger than the depth to which moisture will penetrate during the maximum time parameter τ specified in terminating the solution to that problem.

The boundary condition at the soil surface will be dependent upon the nature of the problem. Should the surface be kept completely wet such that the application rate always equals the intake capacity of the soil, then $\xi = 0$ along this boundary. The value of zero for ξ is obtained from Eq. 2b by noting that under complete saturation the upper limit of the integral equals the lower limit. For application rates less than the intake capacity of the soil, the boundary condition is obtained from Darcy's Law as shown below:

$$v = q = - K_o K_r \frac{\partial h}{\partial y} = - K_o K_r \left(\frac{1}{K_r}\frac{\partial \xi}{\partial y} + 1\right) . \quad . \qquad (8a)$$

or

$$\frac{\partial \xi}{\partial y} = \frac{|q|}{K_o} - K_o \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (8b)$$

The units of q are flux per unit area (or velocity) and the absolute value of q is used in Eq. 8b, so that the value of q may be specified as a positive quantity for an infiltration problem despite the fact that the direction of q is in the negative y direction. The same method of solution and computer program will apply to determining the moisture distribution throughout the soil profile under the action of evapotranspiration. For such applications the value of $|q|/K_o$ would be specified as a negative quantity.

The dimensionless quantity $|q|/K_o$, that is the vertical velocity divided by the saturated hydraulic conductivity, has been used in the computer program as an input specification to define the rate of application. Multiplying this dimensionless quantity by the time parameter $\tau = K_o t$, gives the depth of water which has infiltrated into the soil.

Finite Difference Solution

The initial-boundary value problem depicted in Fig. 3 has been solved by finite differences using the Crank-Nicolson method. (See a text such as Forsythe and Wasow, 1960.) In the Crank-Nicolson method the differences at the advanced time step are weighted equally with those at the current time step. This technique leads to an implicit method which is stable for all incremental time steps (at least for problems with linear partial differential equations in which the coefficients do not vary such as $(1/K_r)(\partial K_r/\partial p)$ do in Eq. 4), and as such requires the solution of a tridiagonal coefficient matrix problem to advance each time step.

The finite difference operator which has been used in the Crank-Nicolson method is obtained by approximating the derivatives in Eq. 4 by second order central differences, and taking the average of these quantaties at the present and advanced time steps. In doing this the derivative with respect to τ is centered midway between adjacent time steps with an incremental spacing equal to one half of $\Delta\tau$. This procedure leads to the following equation:

$$\frac{\xi_{i+1}^{j+1} + \xi_{i-1}^{j+1} - 2\,\xi_i^{j+1}}{\Delta y^2} + \frac{1}{K_r}\,\frac{\partial K_r}{\partial p}\left(\frac{\xi_{i-1}^{j+1} - \xi_{i+1}^{j+1}}{2\Delta y}\right) + \frac{\xi_{i+1}^{j} + \xi_{i-1}^{j} - 2\xi_i^{j}}{\Delta y^2}$$

$$+ \frac{1}{K_r}\,\frac{\partial K_r}{\partial p}\left(\frac{\xi_{i-1}^{j} - \xi_{i+1}^{j}}{2\Delta y}\right) = \frac{2\eta}{K_r}\,\frac{\partial S}{\partial p}\left(\frac{\xi_i^{j+1} - \xi_i^{j}}{\Delta\tau}\right) \quad . \quad . \quad . \quad (9)$$

in which the superscript $j = 1 + \tau/\Delta\tau$ represent the time step and the subscript $i = D/\Delta y - y/\Delta y + 1$ represents the space grid line (note that i has the value unity on the surface and increases with depth below the surface opposite to that of y. In Eq. 9 all quantities with the superscript j are known, whereas quantities with a $j + 1$ superscript are unknown.

Upon placing all unknown quantities on the left and known quantities on the right side of the equal sign, Eq. 9 becomes:

$$\xi_{i-1}^{j+1} - \left(\frac{2+\zeta}{1+\alpha}\right) \xi_i^{j+1} + \left(\frac{1-\alpha}{1+\alpha}\right) \xi_{i+1}^{j+1} = - \xi_{i-1}^{j} - \left(\frac{\zeta - 2}{1+\alpha}\right) \xi_i^{j} - \left(\frac{1-\alpha}{1+\alpha}\right) \xi_{i+1}^{j} \qquad (10)$$

in which

$$\zeta = \frac{2\eta \Delta y^2}{K_r \Delta\tau} \frac{\partial S}{\partial p}$$

and

$$\alpha = \frac{1}{2} \frac{\Delta y}{K_r} \frac{\partial K_r}{\partial p}$$

In Eq. 10 it has been assumed that the coefficients ζ and α resulting from Eq. 9 can all be evaluated from ξ at the present time step and, therefore, K_r, $\partial K_r/\partial p$ and $\frac{\partial S}{\partial p}$ are assumed equal whether they multiply terms with the superscript $j+1$ or j. This approximation should have a relatively small effect on the solution results and was necessary in order to prevent an iterative method of solution in advancing each time step. Table 2 gives the values for these quantities which are needed in the solution of the differential equation and which result from the pressure-saturation data of Table 1.

Finite difference operators for the boundary conditions are obtained similarly. The surface condition which applies for application rates less than the intake capacity of the soil is

$$\xi_0 - \xi_2 = -2\Delta y \left(\frac{|q|}{K_o} - K_r\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

Upon eliminating the nonexistent value ξ_0 by combining Eq. 11 with Eq. 10 results in

$$\left(\frac{2+\zeta}{1+\alpha}\right) \xi_1^{j+1} - \frac{2}{1+\alpha}\, \xi_2^{j+1} = \left(\frac{\zeta-2}{1+\alpha}\right) \xi_1^{j} - 4\Delta y \left(\frac{|q|}{K_o} - K_r\right) + \left(\frac{2}{1+\alpha}\right) \xi_2^{j}$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12a)$$

When the condition is specified which assumes the application rate is equal to the intake capacity of the soil, $\xi_1 = 0$ and it does not change in value. Under these circumstances the first unknown value comes from Eq. 10. Consequently the surface condition finite difference operator for a saturated surface is

$$\left(\frac{2+\zeta}{1+\alpha}\right) \xi_2^{j+1} - \left(\frac{1-\alpha}{1+\alpha}\right) \xi_3^{j+1} = \left(\frac{\zeta-2}{1+\alpha}\right) \xi_2^{j} + \left(\frac{1-\alpha}{1+\alpha}\right) \xi_3^{j} \quad . \; . \; . \; . \; . \quad (12b)$$

The operator for the impervious layer, at which $i = n$ is obtained by the procedure used in obtaining Eq. 12a. This operator is,

$$\xi_{n-1}^{j+1} - \left(\frac{2+\zeta}{2}\right) \xi_n^{j+1} = -\xi_{n-1}^{j} - \left(\frac{\zeta-2}{2}\right) \xi_n^{j} + (1-\alpha)(2\Delta y K_r) \quad . \; . \quad (13)$$

In solving a problem in which no moisture movement exists at $\tau = 0$, it is not necessary to solve for new values of ξ beyond the wetting front because these values will not change. Therefore, in the computer program new values of ξ at the $(j+1)$ time step are computed for only a few increments beyond the wetting front. Values for ξ at this final space increment $i = k$ are obtained from a finite difference operator based on the fact that $\xi_k^{j+1} = \xi_k^{j}$. Using Eq. 10 as the basis of this operator gives,

$$\xi_{k-1}^{j+1} - \left(\frac{2+\zeta}{1+\alpha}\right) \xi_k^{j+1} = - \xi_{k-1}^{j} - \left(\frac{\zeta - 2}{1+\alpha}\right) \xi_i^j - 2\left(\frac{1-\alpha}{1+\alpha}\right) \xi_{k+1}^j \quad \ldots\ldots\ldots\ldots \quad (14)$$

The system of equations which must be solved to advance each new time step results from combining Eqs. 12, 10 and 13 or 14, as the case may be. When written in matrix notation this system becomes

$$A \vec{\xi} = \vec{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

in which

$$A \equiv \begin{bmatrix} a_1 & c_1 & & & & & & \\ 1 & a_2 & c_2 & & & & & \\ 0 & 1 & a_3 & c_3 & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & a_{n-1} & c_{n-1} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & a_n \end{bmatrix}$$

$$\vec{\xi} \equiv \begin{vmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{vmatrix} \quad \text{and} \quad \vec{B} \equiv \begin{vmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{vmatrix}$$

Because of the tridiagonality of the matrix A this system can be solved by a single pass through the rows with a Gaussian elimination bringing the terms equal to unity to zero, after which the unknown values of ξ are computed by back substitution. The implementation of this solution method can be observed by studying the listing of the FORTRAN program in Appendix A.

Nature of Solutions

The solution is obtained in terms of the variable ξ which is introduced in the formulation of the problem through the Kirchhoff transformation. Other quantities, which are better suited for describing the moisture movement, such as the degree of saturation (or moisture content), the hydraulic head and the hydraulic gradient can readily be obtained from the values of ξ at any time-space grid point.

The degree of saturation S is obtained by linear interpolation from the table of values stored in the computer memory (see Table 2). The hydraulic head h is obtained by linearly interpolating between stored values in the table to obtain the pressure head and thereafter adding the elevation of the particular space grid point at which the hydraulic head is being computed. Gradients, if desired, can be computed by dividing the difference between adjacent values by the spacing of the grid network, i.e. Δy. An example of the solution output containing values of ξ, the degree of saturation S and the hydraulic head h is given in Appendix B.

With these quantities available at any number of space increments and time steps, it is possible to determine quantitatively the theoretical movement and behavior of the moisture in the partially saturated soil. Since it is relatively easy and inexpensive to obtain theoretical solutions from the digital computer in comparison with the acquisition of field data, the results from a number of solutions can be analyzed for developing insight into the behavior of the movement under a variety of conditions in various soil types. For example, the solutions to the 12 problems, 23 through 40 in Table 3, required less than one minute of UNIVAC 1108 execution time. The conclusions arrived at from the analysis of the theoretical solutions can become the basis for predicting infiltration and watershed runoff, after a minimum amount of field verification that the theoretical predictions correspond reasonably well to reality. At present, a number of such solutions have been obtained for one soil type

Table 3. Specifications used in describing problems. All problem specifications used the capillary pressure-saturation data in Table 1, plus the following values: η = .394, S_r = .220, λ = 1.110, P_b = .361, Δy = .1, Depth of soil 2 feet, No. of time steps 150 except Prob. No. 19 & 38 where solutions included 55 time steps.

Prob. No.	Initial Head (feet)	Appl. rate $\lvert q\rvert/K_o$	Time Increment $\Delta\tau$
1	-1.0	surface	.005
2	-3.0	saturated	.005
3	-4.0	"	.005
4	-3.0	.159	.005
5	-1.0	.0954	.015
6	-3.0	.0636	.01
7	-3.0	.0477	.01
8	-1.0	.0477	.025
9	-3.0	.0318	.015
10	-2.0	.0318	.02
11	-4.0	.0636	.01
12	-2.0	.0636	.01
13	0.0	.0318	.02
14	0.0	.0636	.01
15	0.0	.0954	.01
16	0.0	.159	.005
17	-1.0	.159	.005
18	-4.0	.159	.005
19	0.0	.2226	.005
20	-2.0	.159	.005
21	-2.0	.2226	.005
23	- .859	.0954	.015
24	-2.859	.0636	.01
25	-2.859	.0477	.01
26	- .859	.0477	.025
27	-2.859	.0318	.015
28	-1.859	.0318	.02
29	-3.859	.0636	.01
30	-1.859	.0636	.01
31	+0.141	.0318	.02
32	+ .141	.0636	.01
33	+ .141	.0954	.01
34	+ .141	.159	.005
35	- .859	.159	.005
37	-3.859	.159	.005
38	.141	.2226	.005
39	-1.859	.159	.005
40	-1.859	.2226	.005

as defined by the saturation pressure data in Table 1. Preliminary analysis of the results of these solutions are included herein for defining influences of the specified initial hydraulic head and the application rate. As additional field data for various soils from the Reynolds Creek Watershed are obtained, additional solutions will be obtained, analysis performed, and the results included in a technical report.

In Fig. 4 the distributions of the degree of saturation with depth has been plotted for several time steps τ from a solution for which the initial hydraulic head h_o was specified equal to -3.0 feet and the application rate $|q|/K_o$ was specified constant and equal to 0.159. (This is Problem No. 4 of Table 3.) From this same solution, the distributions of hydraulic head for several time steps are shown in Fig. 5 and the corresponding gradients of the hydraulic head in Fig. 6.

The quantitative values for saturation, hydraulic head and hydraulic gradient which are given in Figs. 4, 5 and 6, respectively, apply only to the soil with the specified hydraulic properties and for the specified initial condition and constant application rate. Qualitatively, however, the same general pattern of lines of distribution exist for other soils and conditions. It is interesting to note how much greater the hydraulic gradient is near the surface than the unit gradient that exists for saturated flow in a vertical column. Since this latter gradient equals unity, the magnitude of the gradients above unity are due to capillary forces. The magnitude of the relative hydraulic conductivity at the surface at all times must equal the specified $|q|/K_o$ divided by the magnitude of the surface gradient. The magnitude of the surface gradients which are shown for depths equal to zero on Fig. 6 were obtained, however, by differentiation of the polynomial passing through the points defined by values of h at the three grid network points on, and adjacent to, the soil surface.

The change in saturation with the time parameter τ is shown in Fig. 7 for solutions obtained by specifying several rates of application and initial values for the hydraulic head h_o, to correspond to the

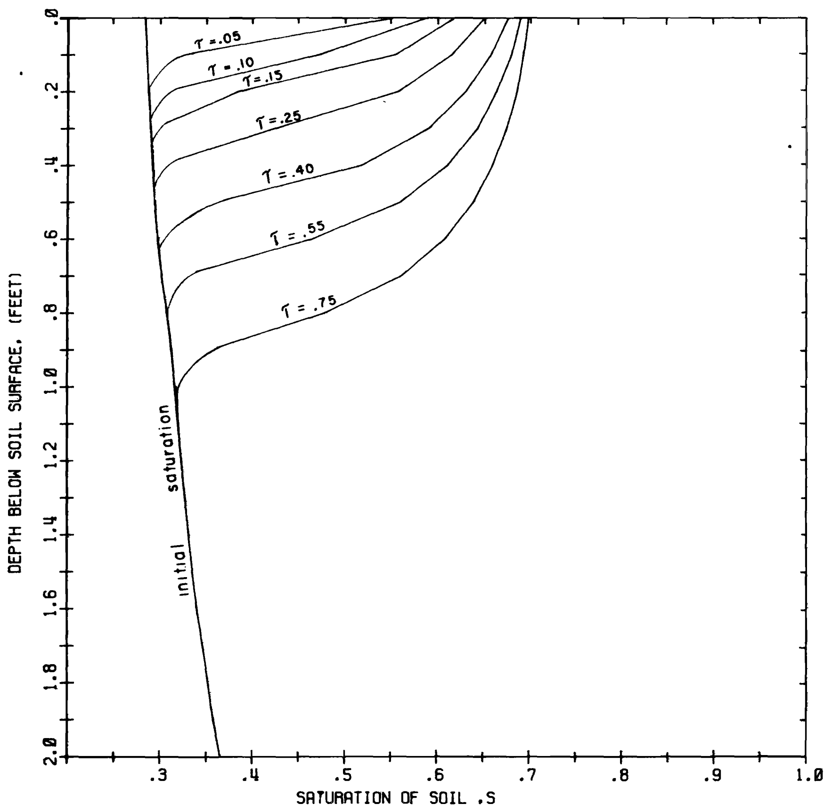


FIG. 4 . DISTRIBUTIONS OF SATURATION WITH DEPTH FOR SEVERAL DIFFERENT VALUES OF THE TIME PARAMETER τ .

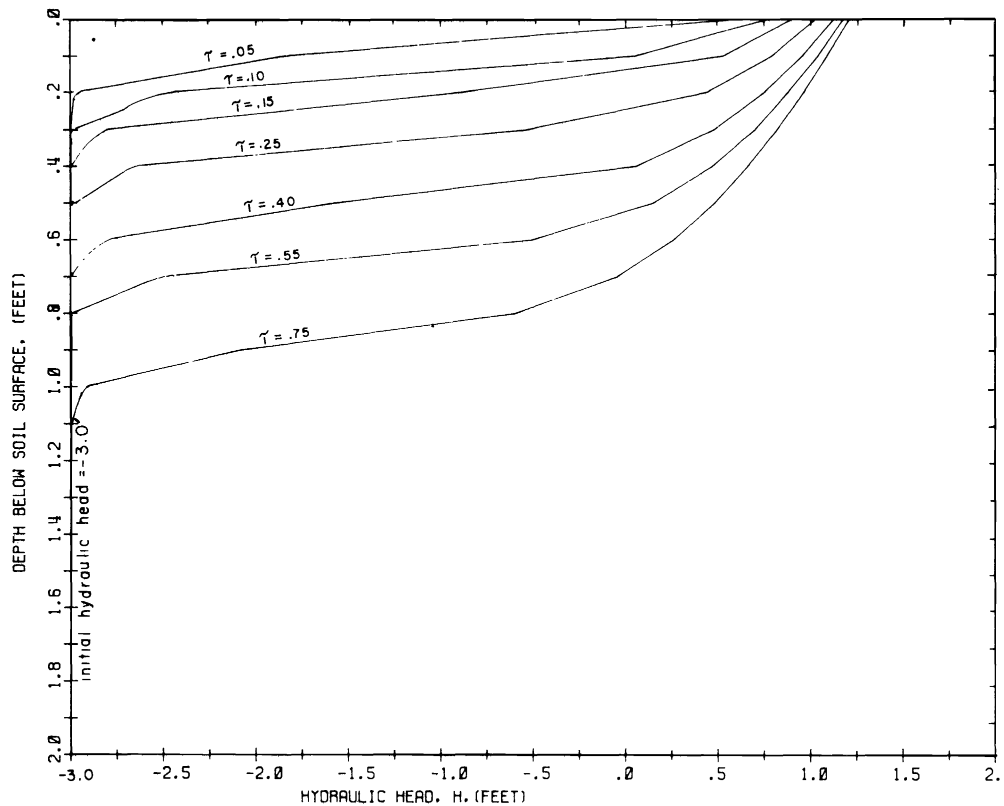


FIG. 5 . DISTRIBUTIONS OF THE HYDRAULIC HEAD WITH DEPTH FOR SEVERAL DIFFERENT VALUES OF THE TIME PARAMETER τ .

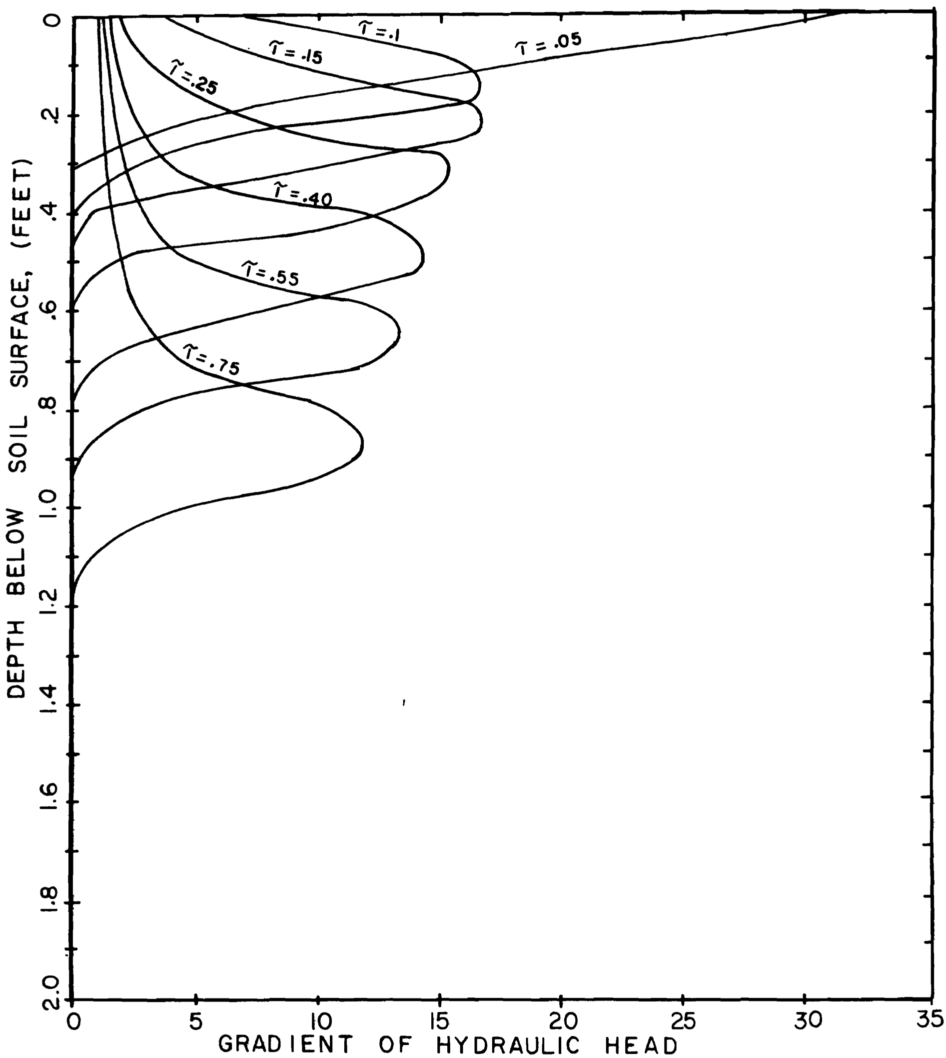
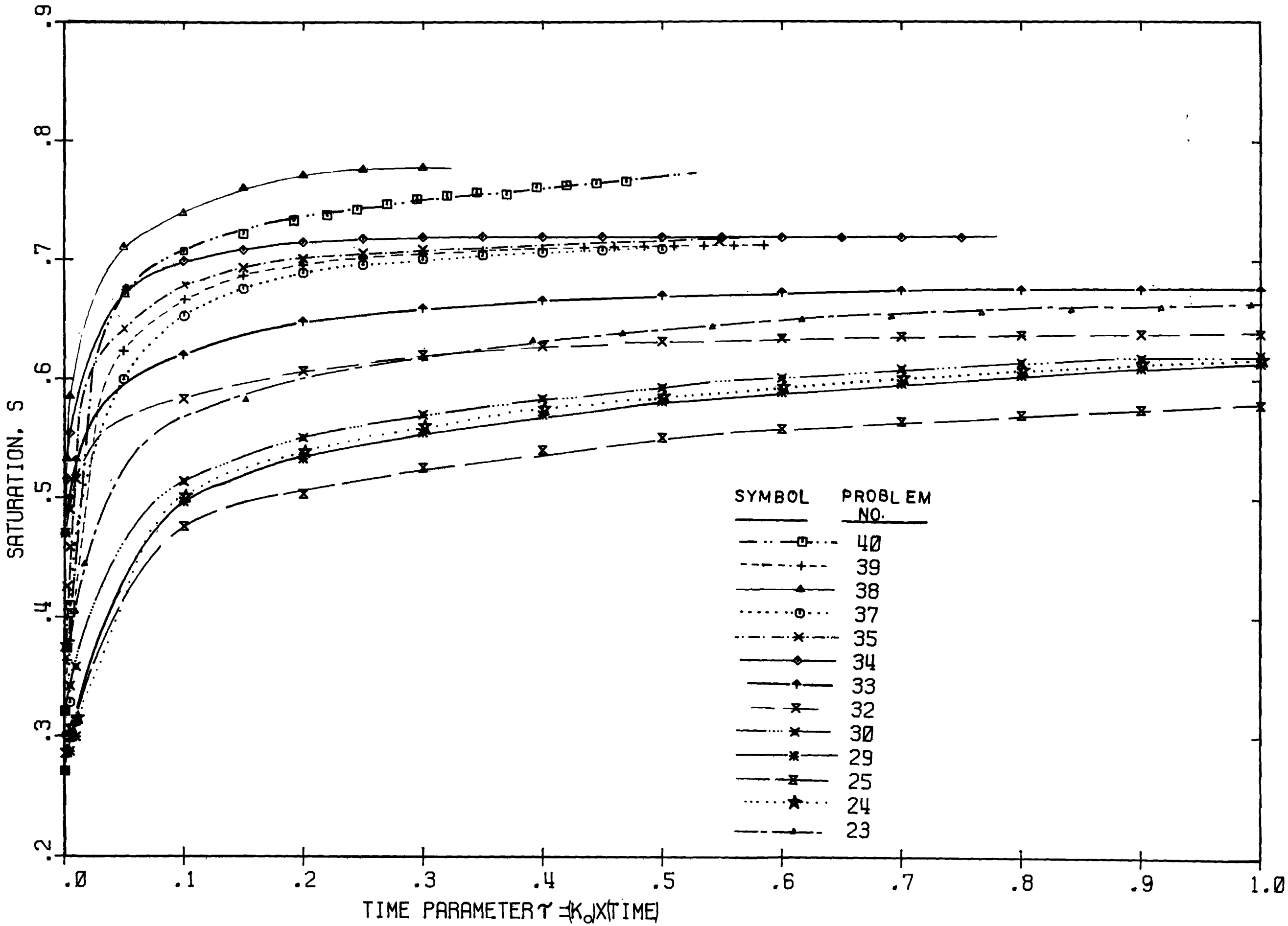


FIG. 6. DISTRIBUTIONS OF THE GRADIENT OF HYDRAULIC HEAD WITH DEPTH FOR SEVERAL VALUES OF THE TIME PARAMETER τ.


SATURATION, S
.2
.3
.4
.5
.6
.7
.8
.9
.Ø
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.Ø
TIME PARAMETER τ =(K$_o$)X(TIME)
SYMBOL
PROBLEM NO.
4Ø
39
38
37
35
34
33
32
3Ø
29
25
24
23

specifications used in obtaining a number of solutions to similarly formulated axisymmetric problems of flow from circular infiltrometers. The different solutions are identified on Fig. 7 by a number associated with each plotting symbol which corresponds to the problem number in Table 3, which contains a summary of the specifications used for each problem. Fig. 8 is a similar plot of the results of saturation at the surface centerline for axisymmetric problems of flow from a circular infiltrometer for which the initial hydraulic heads and application rates per unit area are equal to those for the corresponding one-dimensional problem of Fig. 7 and Table 3. The saturation at the surface centerline of the axisymmetric problems is greater than anywhere else on the soil surface. In fact, the saturation approaches the initial saturation at some distance from the infiltrometer ring, but also decreases within the infiltrometer ring. For all problems, and at all time steps, the magnitude of the saturation for the equivalent axisymmetric problem is less than that for the one-dimensional problem. The decrease in saturation for the axisymmetric case can be attributed to the radial component of velocity removing some water, even at the centerline. The amount of this reduction in saturation is illustrated in Fig. 9, in which the ratios of saturation from the one-dimensional solutions divided by the saturation from the equivalently specified axisymmetric solution have been plotted as ordinate against the time parameter τ as the abscissa.

By noting which problem numbers give the smaller ratios in Fig. 9 and examining the specifications for those problems, it becomes apparent that the radial component is less significant for lower rates of application $|q|/K_o$, particularly if a low rate of application is specified in conjunction with a relatively large value (small in absolute value) of initial hydraulic head h_o.

The influence of the radial component of velocity (or the spreading effect) is different, depending upon the point being considered. The saturation at the surface near the infiltrometer ring will be less than

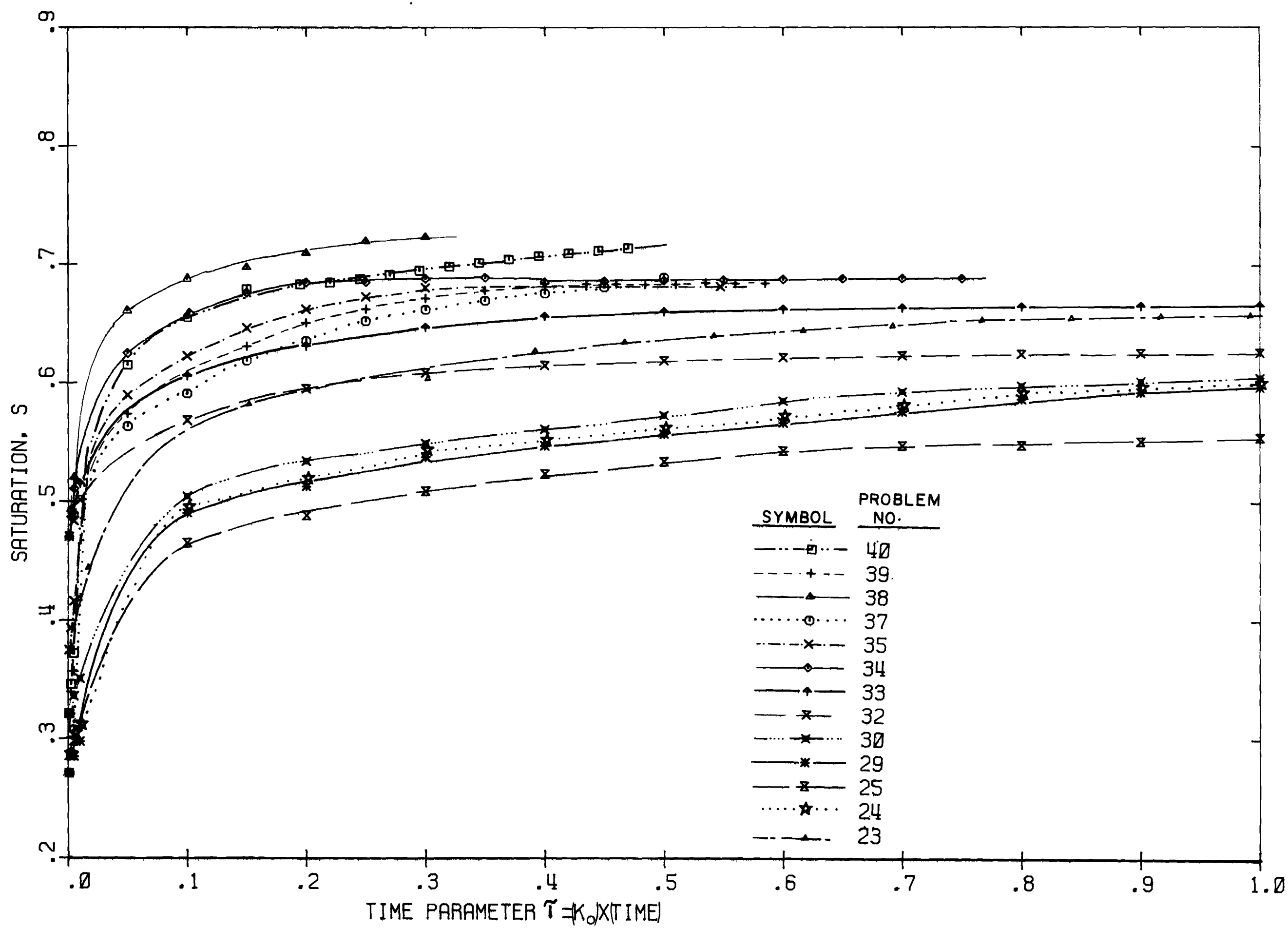


FIG. 8 VARIATION OF SATURATION AT THE

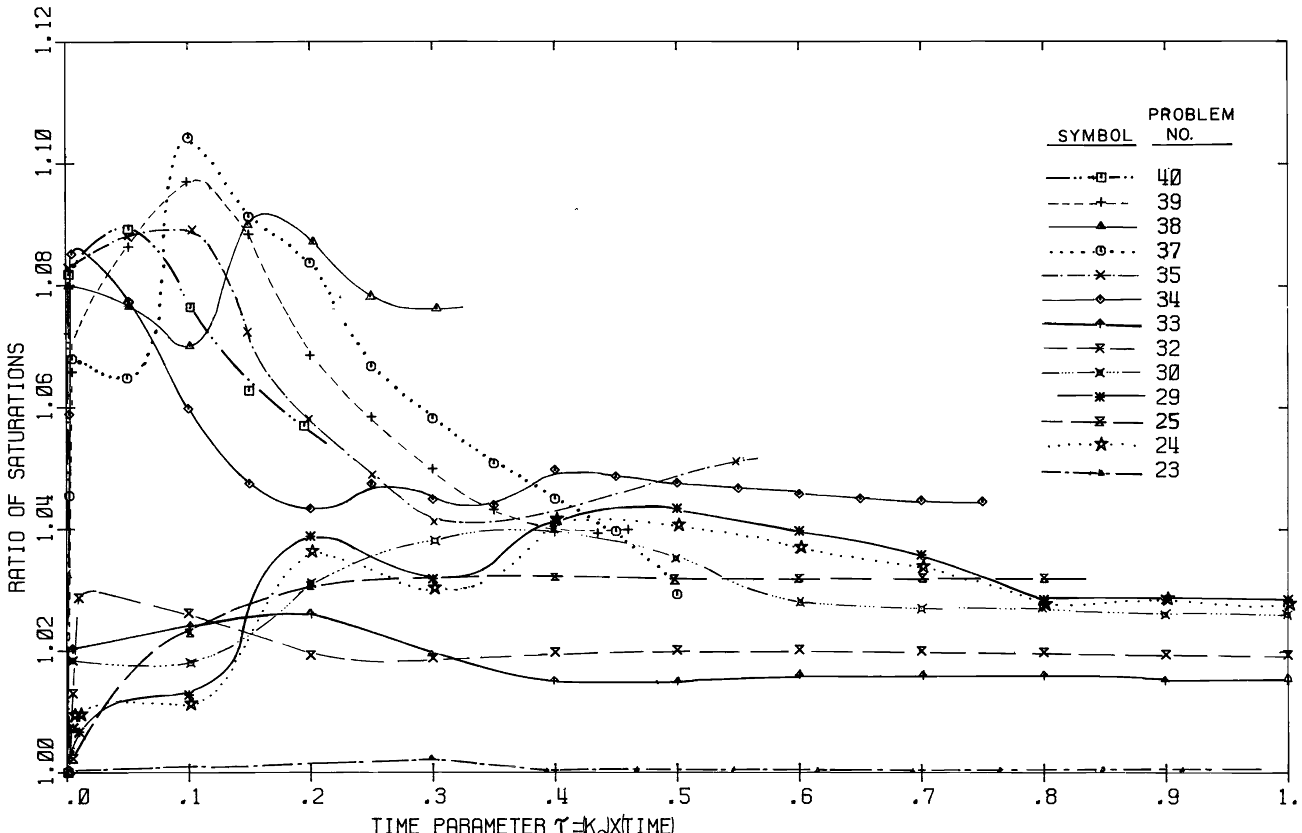


FIG. 9. COMPARISON OF SATURATION AT SURFACE CENTERLINE OF CIRCULAR INFILTROMETER WITH THAT FOR ONLY VERTICAL MOVEMENT.

at the centerline. Table 4 gives values of the ratios between saturation at the centerline and also at the infiltrometer ring for the final value of the time parameter that has been plotted on Fig. 9, as well as the percentage increase in this ratio.

Also of interest is the manner in which the saturation increases at a point within the profile as a function of the application rate as well as the initial condition. The point selected for illustrative purposes is the soil surface for the one-dimensional problem. The values for ratio of saturation at the surface divided by the initial saturation also on the surface have been plotted against the application rate on Fig. 10 for $\tau = 0.1$. Similar ratios, but for $\tau = 0.8$, have been plotted on Fig. 11. On these figures different values of initial head define curves and such curves have also been drawn on these two figures. These figures show how the increase of saturation on the surface increases with smaller values (large in absolute value) of initial hydraulic head and how the magnitude of this increase also increases with increasing rates of application.

For those problems (No. 1 through 4) in which the surface condition has specified that $\xi = 0$, or that the surface be maintained at its maximum saturation, the rate of application will vary with time as the intake capacity of the soil changes. The distribution of saturation and hydraulic head with depth for several time steps are given in Figs. 12 and 13, respectively, using the results from the solution to Problem No. 1. These distributions from Problem No. 1 are qualitatively typical of the distribution of the other 3 problems for which solutions have been obtained.

Table 4. Comparison of saturation at the soil surface from one-dimensional solutions with those at the surface centerline and at the infiltrometer ring from equivalently specified axisyometric problems.

Prob. No.	Initial head h_o	Applic. rate $\|q\|/K_o$	Time Parameter τ	R_o = One-Dim. sat./axisym. sat. at centerline	R_1 = One-Dim. sat./ axisym. sat. at infiltrometer ring (r=1')	Percent diff $\frac{R_1-R_o}{R_o}$ x 100
23	-0.859	0.0954	.99	1.0081	1.0966	9.5
24	-2.859	0.0636	1.0	1.0275	1.1379	10.6
25	-2.859	0.0477	1.0	1.0433	1.1567	10.7
29	-3.859	0.0636	1.0	1.0285	1.1396	10.3
30	-1.859	0.0636	1.0	1.0256	1.1304	10.2
32	+0.141	0.0636	1.0	1.0195	1.0792	5.9
33	+0.141	0.0954	1.0	1.0150	1.0898	7.4
34	+0.141	0.159	.75	1.0446	1.0900	4.3
35	-0.859	0.159	.5475	1.0506	1.1118	5.8
37	-3.859	0.159	.5	1.0293	1.1299	9.8
38	+0.141	-.2226	.3	1.0756	1.1461	6.6
39	-1.859	0.159	.585	1.0415	1.1157	7.1
40	-1.859	0.2226	.470	1.0729	1.2041	12.2

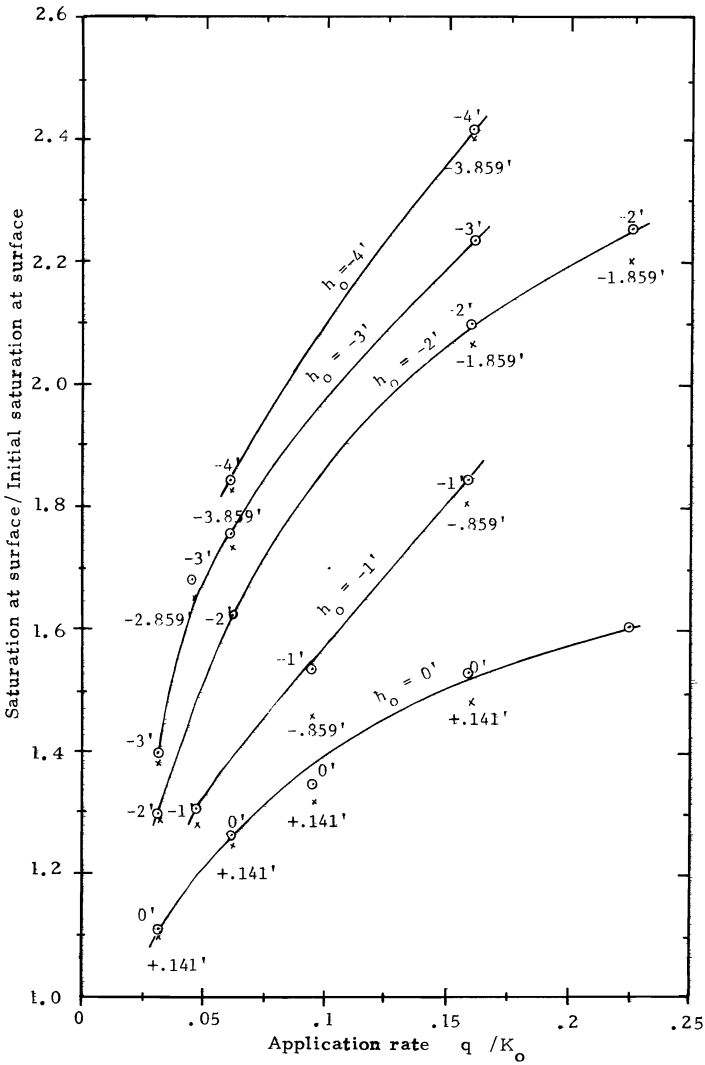


Fig. 10. Variation of the ratio of saturation at the surface to the initial saturation at the surface with the application rate and initial head for the time parameter $\tau = 0.1$.

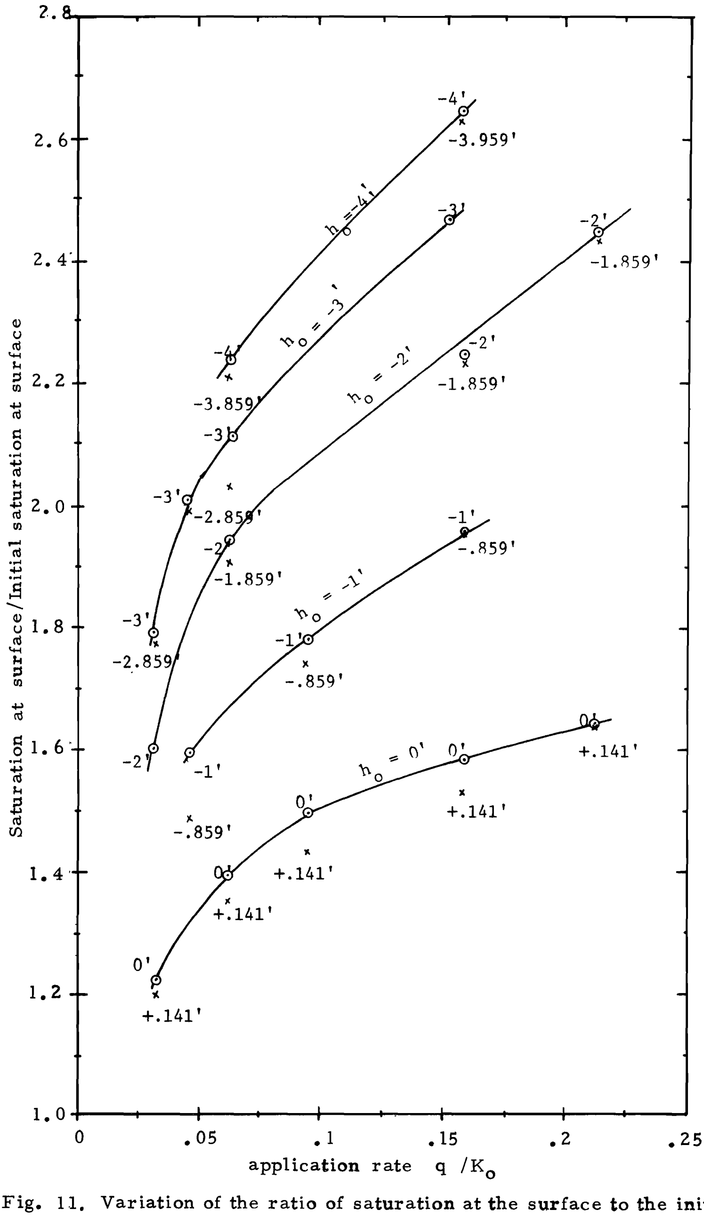


Fig. 11. Variation of the ratio of saturation at the surface to the initial saturation at the surface with the application rate and initial head for the time parameter $\tau = 0.8$.

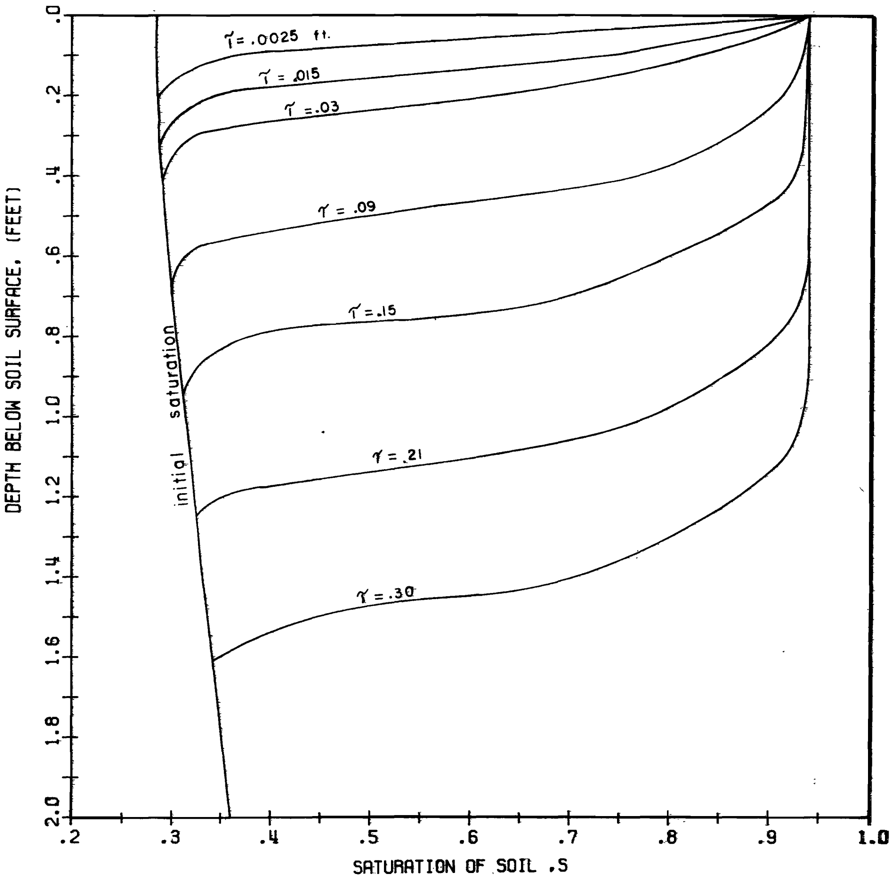


FIG. 12. DISTRIBUTION OF SATURATION WITH DEPTH FOR SEVERAL VALUES OF THE TIME PARAMETER τ WITH THE SURFACE MAINTAINED AT MAXIMUM SATURATION.

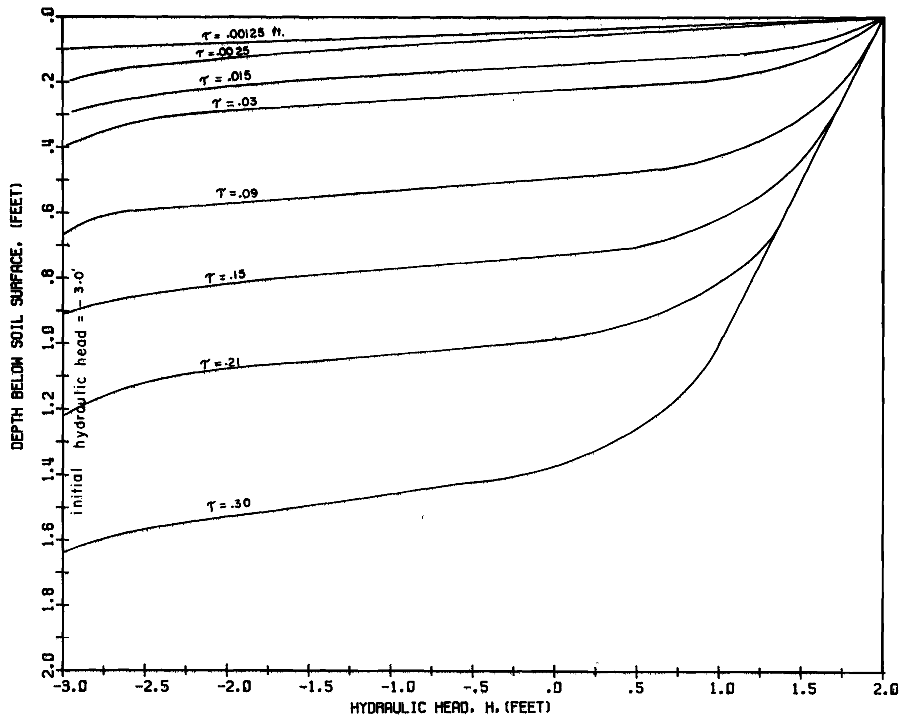


FIG. 13. DISTRIBUTION OF HYDRAULIC HEAD WITH DEPTH FOR SEVERAL VALUES OF THE TIME PARAMETER τ WITH THE SURFACE MAINTAINED AT MAXIMUM SATURATION.

Conclusions

Numerical solutions to mathematical problems which describe moisture movement in partially saturated soils provide an economical means of studying a number of aspects of the flow phenomena. In order for the conclusions which might be drawn from these studies to have physical significance, the mathematical formulation must contain no assumptions which are not in accordance with the physical processes which occur. In formulating the problem which is described herein, of vertical moisture movement resulting from application of water at the soil surface, a fundamentally sound approach has been used, which incorporates conservation of mass, energy and momentum on a macro scale, through the continuity equation of fluid dynamics and Darcy' s Law, with the best means presently available for quantitatively describing the hydraulic properties of partially saturated soils under imbibition. In other words, the formulation has combined fundamental theory with results from past experimental studies which have defined important behavioral characteristics under specially controlled conditions. Despite this a final verification of the mathematical solution results is needed before one can be fully confident that the theoretical solution results corresponds quantitatively to the physical flow characteristics. Experiments have been designed by the Agricultural Research Service to duplicate the mathematical problem of flow from a circular infiltrometer in which measurements of saturations, capillary pressures and fluxes will be obtained at a number of points throughout the soil. Final verficiation (or lack thereof) will be made after the experiments are completed. Even in the absence of such final verification the following conclusions can be drawn from this study: (Some of these conclusions reinforce conclusions arrived at by other studies).

1. With presently available high speed digital computer solutions to partially saturated flow problems can be obtained very economically. Solutions covering a large number of time steps, of vertical moisture movement discussed in this paper can be obtained for less than one

dollar of execution time (page costs and other I/O charges must be added).

2. A large number of characteristics of the flow pattern can be examined from numerical solutions. In addition sensitivity analyses can be carried out to determine the effects on the flow pattern of variations in problem specifications by studying the results from a number of problems. Likewise, the influences of various soil properties on such aspects as rate of penetration, increases in saturation, magnitude and distribution of gradients and flux can be determined.

3. The Burdine Theory when modified slightly to prevent division by zero provides a method for obtaining hydraulic conductivity from saturation - capillary pressure data for natural soil and not just for ordered porous media. This conclusion is based on the fact that good agreement exists from the theory and measured values for the soil used in this study, and there is no reason to suspect poorer agreement for other soils.

4. Numerical solution can sharply define the location of the wetting front.

5. Boundary effects on circular infiltrometers significantly alter the flow pattern, even reducing the saturation at the surface centerline appreciably over that which would exist for the same application rate over an infinite area.

6. Hydraulic gradients due to capillary forces are much greater in magnitude than gravitational forces in the infiltration problem and, consequently, are very important in determining flow characteristics.

References

Brooks, R. H. and A. T. Corey (1964) "Hydraulic Properties of Porous Media," Hydrology Paper No. 3, Colorado State University, Fort Collins, Colorado.

Brooks, R. H. and A. T. Corey (1966) "Properties of Porous Media Affecting Fluid Flow," Jour. of the Irrigation and Drainage Division, ASCE, Vol. 92, No. IR2, Proc. Paper 4855, June, pp. 61-88.

Burdine, M. T. (1952) "Relative Permeability Calculations from Pore-Size Distribution Data," Transactions, AIME.

Forsythe, G. E. and W. R. Wasow (1960) Finite Difference Methods for Partial Differential Equations, John Wiley & Sons, Inc., New York.

Jeppson, R. W. (1970a) "Transient Flow of Water from Infiltrometers--Formulation of Mathematical Model and Preliminary Numerical Solutions and Analyses of Results," Report PRWG-59c-2, Utah Water Research Laboratory, Utah State University, Logan, June.

Jeppson, R. W. (1970b) "Formulation and Solution of Transient Flow of Water from an Infiltrometer Using the Kirchhoff Transformation," Report PRWG-59c-3, Utah Water Research Laboratory, Utah State University, Logan, July.

Jeppson, R. W. (1970c) "Determination of Hydraulic Conductivity-Capillary Pressure Relationship from Saturation-Capillary Pressure Data from Soils," Report PRWG-59c-4, Utah Water Research Laboratory, Utah State University, Logan, July.

King, L. G. (1965) "Description of Soil Characteristics for Partially Saturated Flow," Soil Science Society of America, Proceedings, Vol. 29, No. 4, July-August, pp. 359-362.

Laliberte, G. E., R. H. Brooks and A. T. Corey (1968) "Permeability Calculated from Desaturated Data," Jour. of the Irrigation and Drainage Division, ASCE, Vol. 94, No. IR1, Proc. Paper 5843, March, pp. 57-71.

Renger, Von M., W. Giesel, O. Strebel and S. Lorch (1970) "Erste Ergebnisse zur quantitativen Erfassung der Wasserhaushaltskomponenten in der ungesättigten Bodenzone," Zeitschrift fur Pflanzenernährung und Bodenkunde, Verlag Chemie, GmbH., Weinheim/Bergstr., 126. Band, Heft 1, pp. 15-35.

White, M. F., H. R. Duke, D. K. Sunada and A. T. Corey (1970) "Physics of Desaturation in Porous Materials," Jour. of the Irrigation and Drainage Division, ASCE, Vol. 96, No. IR2, Proc. Paper 7376, June, pp. 165-191.

APPENDIX A. FORTRAN listing of computer program for solving one-dimensional moisture movement from the soil surface using a modified Burdine Theory and a table look-up technique.

```
@FOR,I   ONEDM,ONEDM
      INTEGER IWRT(20)
      COMMON XI(50),XI1(32),A(50),DSP(50),DKP(50),SAT(50),RK(50),ST(32),
     $       B(32),H(32),Z2(50),C(50),SR,PB,POR,AMBDA,DELT,DELS,DELT1,
     $DELT2,HC,PINB,NY,NY1,MY,MY1,NCB,NINC,NCBP
   86 READ(5,100) MY,NT,NTAB,HEIGT,DELT,Q,APPL
C IF SURFACE SATURATED THEN APPL .GE. 1.0 AND Q REPRESENTS MAXI. SATURATION
      IF(MY .GT. 32) STOP
  100 FORMAT(3I5,6F10.5)
      READ(5,100) INCB,NRT,NXI,SR,AMBDA,POR,PB,HC,PI
      READ(5,130) INR,(IWRT(I),I=1,INR)
  130 FORMAT(16I5)
      WRITE(6,101) MY,NT,HEIGT,DELT,Q,NRT,NXI,SR,AMBDA,POR,PB,HC,PI,INCB
  101 FORMAT('1MY=',I3,' NT='I4,' DEPTH=',F6.2,' DELT=',E10.5,' Q=',F8.5
     $,/ ' NRT=',I4,' NXI=',I3,' SR=',F6.3,' LAMBDA=',F6.3,' POROSITY=',
     $F6.3,' PB=',F6.3,' HC=',E10.5,' PI=',F7.2,' INCB=',I3)
      NY=7
      NY1=6
      NY4=3
      MY1=MY-1
      TIM1=0.0
      WATE1=0.0
      WRATE=0.0
      DELS=HEIGT/FLOAT(MY1)
      IF(INCB .EQ. 0) GO TO 37
      DET=DELT
      FDT=1.0/FLOAT(INCB)
      DELT=DELT*FDT
      NTR=NRT
      NRT=1
      IF(NTAB .GT. 0) GO TO 37
      DO 42 I=1,NINC
   42 DSP(I)=DSP(I)/FDT
   37 DEL4=4.*DELS
      DEL2=2.*DELS
      IF(NTAB .GT. 0) CALL TABLOK
C INITIALIZATION
      PFIR=PINB+DELT1*FLOAT(NCB)
      DO 1 I=1,MY
      P=HEIGT-DELS*FLOAT(I-1)-PI
      IF(P-PFIR) 4,5,5
    4 FI=(P-PINB)/DELT1+1.0
      II=FI
      FINT=FI-FLOAT(II)
      XI1(I)=XI(II)+FINT*(XI(II+1)-XI(II))
      ST(I)=SAT(II)+FINT*(SAT(II+1)-SAT(II))
      GO TO 1
    5 FI=(P-PFIR)/DELT2
      IJ=FI
      II=IJ+NCBP
      IF(II .GE. NINC) II=NINC-1
      FINT=FI-FLOAT(IJ)
      XI1(I)=XI(II)+FINT*(XI(II+1)-XI(II))
      ST(I)=SAT(II)+FINT*(SAT(II+1)-SAT(II))
    1 Z2(I)=XI1(I)
      N1=1
```

```
      N2=2
      IF(APPL .LT. 1.0) GO TO 43
      Z2(1)=0.0
      XI1(1)=0.0
      H(1)=HEIGT
      ST(1)=Q
      N1=2
      N2=3
   43 IF(NXT .EQ. 0) GO TO 2
      WRITE(6,102) (XI1(I),I=1,MY)
  102 FORMAT(' INITIAL XI', 10(/,1H ,16F8.4))
    2 CALL MOIST(WATER,0)
      WRITE(6,103) WATER,(ST(I),I=1,MY)
  103 FORMAT(' WATER CONTENT=',F10.5,' INITIAL SATURATION',10(/,1H ,
     $ 16F8.4))
C  ESTABLISHMENT OF SYSTEM OF EQUATIONS
      IF(INR .EQ. 0) GO TO 45
      IIWR=1
      CALL INOUT(0,9,ST(1),MY)
   45 MK=NINC-3
      TIME=0.0
      J=1
    3 J1=J-1
      NJP=MOD(J1,NRT)
      IF(J1 .LT. INCB) NJP=0
      IF(APPL .GT. .9) GO TO 44
    7 IF(XI(MK) .GT. XI1(1) .OR. MK .EQ. NINC) GO TO 6
      MK=MK+1
      GO TO 7
    6 IF(XI(MK-1) .LT. XI1(1) .OR. MK .EQ. 2) GO TO 8
      MK=MK-1
      GO TO 6
    8 MKM=MK-1
      FINT=(XI1(1)-XI(MKM))/(XI(MK)-XI(MKM))
      ZETA=DSP(MKM)+FINT*(DSP(MK)-DSP(MKM))
      ALTA=DKP(MKM)+FINT*(DKP(MK)-DKP(MKM))
      ALTA1=ALTA+1.
      C(1)=-2./ALTA1
      A(1)=(ZETA+2.)/ALTA1
      IF(NJP .GT. 0) GO TO 21
      ST(1)=SAT(MKM)+FINT*(SAT(MK)-SAT(MKM))
      IF(MKM .GT. NCB) GO TO 20
      H(1)=HEIGT-PINB-DELT1*(FLOAT(MKM-1) +FINT)
      GO TO 21
   20 H(1)=HEIGT-PFIR-DELT2*(FLOAT(MKM-NCBP)+FINT)
   21 RKK=RK(MKM)+FINT*(RK(MK)-RK(MKM))
      B(1)=2.*XI1(2)/ALTA1+DEL4*(RKK-Q)+(ZETA-2.)*XI1(1)/ALTA1
   44 DO 9 I= 2,NY1
   10 IF(XI(MK) .GT. XI1(I) .OR. MK .EQ. NINC) GO TO 11
      MK=MK+1
      GO TO 10
   11 IF(XI(MK-1) .LT. XI1(I) .OR. MK .EQ. 2) GO TO 12
      MK=MK-1
      GO TO 11
   12 MKM=MK-1
      FINT=(XI1(I)-XI(MKM))/(XI(MK)-XI(MKM))
      ZETA=DSP(MKM)+FINT*(DSP(MK)-DSP(MKM))
      ALTA=DKP(MKM)+FINT*(DKP(MK)-DKP(MKM))
      ALTA1=ALTA+1.
      C(I)=(1.0-ALTA)/ALTA1
      ZE2=(2.-ZETA)/ALTA1
      A(I)=-(ZETA+2.)/ALTA1
      IF(NJP .GT. 0) GO TO 9
```

```
      ST(I)=SAT(MKM)+FINT*(SAT(MK)-SAT(MKM))
      IF(MK.GT. NCB) GO TO 22
      H(I)=HEIGT-DELS*FLOAT(I-1)-PINB-DELT1*(FLOAT(MKM-1)+FINT)
      GO TO 9
   22 H(I)=HEIGT-DELS*FLOAT(I-1)-PFIR-DELT2*(FLOAT(MKM-NCBP)+FINT)
    9 B(I)=ZE2*XI1(I)-XI1(I-1)-C(I)*XI1(I+1)
   13 IF(XI(MK) .GT. XI1(NY) .OR. MK .EQ. NINC) GO TO 14
      MK=MK+1
      GO TO 13
   14 IF(XI(MK-1) .LT. XI1(NY) .OR. MK .EQ. 2) GO TO 15
      MK=MK-1
      GO TO 14
   15 MKM=MK-1
      FINT=(XI1(NY)-XI(MKM))/(XI(MK)-XI(MKM))
      ZETA=DSP(MKM)+FINT*(DSP(MK)-DSP(MKM))
      ALTA=DKP(MKM)+FINT*(DKP(MK)-DKP(MKM))
      IF(NJP .GT. 0) GO TO 24
      ST(NY)=SAT(MKM)+FINT*(SAT(MK)-SAT(MKM))
      IF(MK .GT. NCB) GO TO 23
      H(NY)=HEIGT-DELS*FLOAT(NY1)-PINB-DELT1*(FLOAT(MKM-1)+FINT)
      GO TO 24
   23 H(NY)=HEIGT-DELS*FLOAT(NY1)-PFIR-DELT2*(FLOAT(MKM-NCBP)+FINT)
   24 IF(NY .LT. MY) GO TO 31
      A(NY)=-.5*(2.+ZETA)
      ZE2=-.5*(ZETA-2.)
      RKK=RK(MKM)+FINT*(RK(MK)-RK(MKM))
      B(NY)=-XI1(NY1)+DEL2*RKK*(1.-ALTA)+ZE2*XI1(NY)
      GO TO 32
   31 ALTA1=ALTA+1.
      A(NY)=-(ZETA+2.)/ALTA1
      ZE2=(2.-ZETA)/ALTA1
      B(NY)=ZE2*XI1(NY)-XI1(NY1)-2.*(1.-ALTA)*XI1(NY+1)/ALTA1
   32 IF(NJP .GT. 0) GO TO 36
      IF(J1 .EQ. 0) GO TO 47
      CALL MOIST(WATER,J1)
      WRATE=(WATER-WATE1)/(TIME-TIM1)
   47 WRITE(6,105) WATER,WRATE,J1,TIME,(ST(I),I=1,NY)
  105 FORMAT(' INCREASED WATER=',F10.5,' RATE=',F10.5,' SATURATION FOR S
     $TEP',I5,' TAU=',E12.6,10(/,1H ,16F8.4))
      WATE1=WATER
      TIM1=TIME
      WRITE(6,106) J1,TIME, (H(I),I=1,NY)
  106 FORMAT(' HYDRAULIC HEAD FOR STEP',I5,' TAU=',E12.6,10(/,1H ,16F8.3
     $))
      IF(INR .EQ. 0) GO TO 36
      IF(MOD(J1,IWRT(IIWR)).NE. 0) GO TO 36
      IIWR=IIWR+1
      WRITE(6,138) J1,NY
  138 FORMAT(' SATURATION AND HEAD FOR TIME STEP',I4,' HAVE BEEN WRITTEN
     $ ,NY=',I5)
      CALL INOUT(0,9,ST(1),NY)
      CALL INOUT(0,9,H(1),NY)
C  SOLVING SYSTEM OF EQUAITONS
   36 DO 33 I=N2,NY
      I1=I-1
      A(I)=A(I)-C(I1)/A(I1)
   33 B(I)=B(I)-B(I1)/A(I1)
      I=NY
      XI1(I)=B(I)/A(I)
   34 IP=I
      I=I-1
      XI1(I)=(B(I)-C(I)*XI1(IP))/A(I)
      IF(I .GT. N1) GO TO 34
```

```
      TIME=TIME+DELT
C PRINTING OF RESULTS
      IF(NXI .EQ. 0 .OR. MOD(J,NRT) .GT. 0) GO TO 30
      WRITE(6,104) J,TIME,(XI1(I),I=1,NY)
  104 FORMAT(' XI FOR STEP',I4,' TAU=TIME X KO=',E12.5,10(/,1H ,16F8.4))
   30 IF(XI1(NY4) .GT. Z2(NY4)-HC .OR. NY .EQ. MY) GO TO 40
      NY1=NY
      NY=NY+1
      NY4=NY-4
      GO TO 30
   40 IF(INCB .EQ. 0) GO TO 38
      J=J+1
      IF(J .LE. INCB) GO TO 3
      INCB=0
      DELT=DET
      NRT=NTR
      DO 39 I=1,NINC
   39 DSP(I)=FDT*DSP(I)
      J=1
   38 J=J+1
      IF(J .LE. NT) GO TO 3
      GO TO 86
      END
@FOR,I   DETWAT,DETWAT
      SUBROUTINE MOIST(WATER,J1)
      REAL SST(32)
      COMMON XI(50),XI1(32),A(50),DSP(50),DKP(50),SAT(50),RK(50),ST(32),
     $       B(32),H(32),Z2(50),C(50), SR,PB,POR,AMBDA,DELT,DELS,DELT1,
     $DELT2,HC,PINB,NY,NY1,MY,MY1,NCB,NINC,NCBP
      IF(J1 .GT. 0) GO TO 1
      SST(1)=ST(1)
      DO 2 I=2,MY
    2 SST(I)=ST(I)+SST(I-1)
      FAC=POR*DELS
      WATER=FAC*SST(MY)
      RETURN
    1 SUM=ST(1)
      DO 3 I=2,NY
    3 SUM=SUM+ST(I)
      WATER=FAC*(SUM-SST(NY))
      RETURN
      END
@FOR,I   TABK1,TABK1
      SUBROUTINE TABLOK
      COMMON XI(50),XI1(32),R(50),DSP(50),DKP(50),SAT(50),RK(50),ST(32),
     $       Z(32),H(32),PC(50),S(50), SR,PB,POR,AMBDA,DELT,DELS,DELT1,
     $DELT2,HC,PINB,NY,NY1,MY,MY1,NCB,NINC,NCBP
      NINC=50
   10 READ(5,100) N,NFT,PO,PINBC,NINBC,SAT1
  100 FORMAT(2I5,2F10.5,I5,2F10.5)
      POO=PO
      WRITE(6,102) AMBDA,SR,PB,PO,PINBC,NINBC,SAT1
  102 FORMAT(' LAMBDA=',F8.3,' SR=',F8.4,' PB=',F8.3,' PO=',F8.4,' PINBC
     $=',F8.2,' NINBC=',I5,' SAT1=',F8.4)
      UNITY=-1.0
      N1=N-1
      READ(5,101) (PC(I),S(I),I=1,N)
  101 FORMAT(8F10.5)
      IF(NFT .GT. 0) GO TO 12
      FTCM=1.0/30.4801
      DO 13 I=1,N
   13 PC(I)=FTCM*PC(I)
   12 WRITE(6,107) (I,PC(I),S(I),I=1,N)
```

```
      PINB=PC(1)
      IF(NINBC .EQ. 0) GO TO 14
      DO 287 I=1,N
  287 PC(I)=PC(I)+PINBC
   14 I1=1
  107 FORMAT(' CAPILLARY TENSION AND SATURATION INPUT DATA',20(/,1H , 3(
     $I3,2F10.5)))
      I2=2
      I3=3
      NC3=NINC/3
      NCB=2*NC3
      NCF=NINC-NCB
      DELT1=(PC(N)-PC(I1))/FLOAT(NCB+2*(NCF-1))
      DELT2=2.*DELT1
      NCBP=NCB+1
      SRM1=SAT1-SR
      ALPHA=3.*AMBDA+2.
      P=PC(1)
      SUM1=0.0
      C1=PC(I1)/((S(I1)-S(I2))*(S(I1)-S(I3)))
      C2=PC(I2)/((S(I2)-S(I1))*(S(I2)-S(I3)))
      C3=PC(I3)/((S(I3)-S(I1))*(S(I3)-S(I2)))
      C=C1+C2+C3
      C5=2.*C
      CCC=C1*(S(I2)+S(I3))+C2*(S(I1)+S(I3))+C3*(S(I1)+S(I2))
      A=C1*S(I2)*S(I3)+C2*S(I1)*S(I3)+C3*S(I1)*S(I2)
      B=-CCC
      DELTAP=DELT1
      DO 1 I=1,NINC
      IF(P .LT. PC(I2) .OR. I2 .EQ. N1) GO TO 3
    2 I1=I1+1
      I2=I2+1
      I3=I3+1
      IF(P .GT. PC(I2) .AND. I2 .LT. N1) GO TO 2
      C1=PC(I1)/((S(I1)-S(I2))*(S(I1)-S(I3)))
      C2=PC(I2)/((S(I2)-S(I1))*(S(I2)-S(I3)))
      C3=PC(I3)/((S(I3)-S(I1))*(S(I3)-S(I2)))
      C=C1+C2+C3
      C5=2.*C
      CCC=C1*(S(I2)+S(I3))+C2*(S(I1)+S(I3))+C3*(S(I1)+S(I2))
      A=C1*S(I2)*S(I3)+C2*S(I1)*S(I3)+C3*S(I1)*S(I2)
      B=-CCC
    3 IF(I .EQ. NCBP) DELTAP=DELT2
      P2=P+DELTAP
      BB4A=B*B-4.*(A-P)*C
      SBB4A=DSQRT(BB4A)
      SAT2=(CCC-SBB4A)/(2.*C)
      SAT22=(CCC+SBB4A)/(2.*C)
      SLT22=S(I2)+(S(I1)-S(I2))*(P-PC(I2))/(PC(I1)-PC(I2))
      IF(ABS(SAT22-SLT22) .LT. ABS(SAT2-SLT22)) SAT2=SAT22
      R(I)= (SAT2-SR)/SRM1
      Q=4.*A*C-B*B
      SM0=C5*SAT1+B
      SM=C5*SAT2+B
      SAT(I)=SAT2
      DSP(I)=UNITY/SM
      SQQ=SQRT(ABS(Q))
      IF(I .LT. NINC) GO TO 86
      SE1=(SAT1-SR)/SRM1
      SE2=(SAT2-SR)/SRM1
      AB2=(2.+AMBDA)/AMBDA
      SPBA=SRM1/(4.*AB2)
      SM1=SPBA*SE1**AB2
```

```
      SUM=SPBA*SE2**AB2
      SUM1=SUM1+SM1-SUM
      RK(I)=SUM1
      SUM1=SUM1+SUM
      GO TO 85
86    SM2=SM/(Q*P)+4.*C/(Q*SQQ)*ATAN2(SM,SQQ)
      SM1=SM0/(Q*P0)+4.*C/(Q*SQQ)*ATAN2(SM0,SQQ)
      SUM1=SUM1+ABS(SM1-SM2)
      RK(I)=SUM1
      SAT1=SAT2
      P0=P
 85   DKP(I)=P
  1   P=P2
      P=PINB
      P=PC(1)
      DELTAP=DELT1
      DO 4 I=1,NINC
      RK(I)=SUM1-RK(I)
      RR=R(I)**2
      RRK=RR*RK(I)/SUM1
      DKSE=(RR*SRM1/DKP(I)**2+2.*RK(I)*R(I))/SUM1
      DKP(I)=DSP(I)*DKSE/(SRM1*RRK)
      RK(I)=RRK
      DSP(I)=DSP(I)/RRK
      IF(I .EQ. NCBP) DELTAP=DELT2
      Z(I)=P-PINBC
  4   P=P+DELTAP
      XI(1)=.5*(1.0+RK(1))*(PC(1)-P00)
      DP13=.54166667*DELT1
      DP124=.041666667*DELT1
      XI(2)=XI(1)+.5*DELT1*(RK(1)+RK(2))
      NINC1=NINC-1
      DO 15 I=3,NINC1
      IF(I .NE. NCBP) GO TO 15
      DP13=.54166667*DELT2
      DP124=.041666667*DELT2
 15   XI(I)=XI(I-1)+DP13*(RK(I)+RK(I-1))-DP124*(RK(I+1)+RK(I-2))
      XI(NINC)=XI(NINC1)+.5*DELTAP*(RK(NINC)+RK(NINC1))
      WRITE(6,123)
123   FORMAT('0 VALUES OF P,SATURATION,XI,DS/DP/DR, (DK/KP/KR) AND KR
     $ DETERMINED FROM S VS PC DATA')
      NHNIC=NINC/2
      NNTC=NHNIC
       IF(NINC-NNTC .GT. NNTC) NNTC=NINC-NNTC
      FAC=2.*POR*DELS*DELS/DELT
      FAC1=.5*DELS
      DO 6 I=1,NNTC
      II=NHNIC+I
      IF(II .GT. NINC) II=NINC
      WRITE(6,125) I,,Z(I),SAT(I),XI(I),DSP(I),DKP(I),RK(I),II,,Z(II),
     $SAT(II),XI(II),DSP(II),DKP(II),RK(II)
125   FORMAT(1H ,I2,2F8.4,4E11.5,I7,2F8.4,4E11.5)
      DSP(I)=FAC*DSP(I)
      DSP(II)=FAC*DSP(II)
      DKP(I)=FAC1*DKP(I)
  6   DKP(II)=FAC1*DKP(II)
      RETURN
      END
```

APPENDIX B. Sample of solution output including values of ξ, saturation and hydraulic head at a number of time steps.

```
 INITIAL XI
   .3065   .3065   .3065   .3065   .3065   .3065   .3065   .3065   .3065   .3065   .3065   .3064
   .3064   .3064   .3064   .3063   .3063   .3062   .3062   .3061   .3060
  INITIAL SATURATION
   .2832   .2851   .2869   .2892   .2916   .2939   .2972   .3021   .3078   .3128   .3163   .3196
   .3231   .3269   .3310   .3354   .3404   .3463   .3523   .3582   .3652
 SATURATION FOR STEP    0 TAU=  .000000
   .2832   .2851   .2869   .2892   .2916   .2939   .2972
 HYDRAULIC HEAD FOR STEP    0 TAU=  .000000
  -3.000  -3.000  -3.000  -3.000  -3.000  -2.997  -2.897
 XI FOR STEP    1 TAU=TIME X KO=   .25000-02
   .3065   .3065   .3065   .3065   .3065   .3065   .3065
  SATURATION FOR STEP    1 TAU=  .250000-02
   .2977   .2851   .2869   .2892   .2916   .2940   .2972
 HYDRAULIC HEAD FOR STEP    1 TAU=  .250000-02
  -2.386  -2.999  -3.000  -3.000  -3.000  -2.996  -2.897
 XI FOR STEP    2 TAU=TIME X KO=   .50000-02
   .3065   .3065   .3065   .3065   .3065   .3065   .3065
 XI FOR STEP   10 TAU=TIME X KO=   .50000-01
   .2954   .3064   .3065   .3065   .3065   .3065   .3065
 SATURATION FOR STEP   10 TAU=  .500000-01
   .5543   .3247   .2873   .2892   .2916   .2940   .2972
 HYDRAULIC HEAD FOR STEP   10 TAU=  .500000-01
    .627  -1.858  -2.979  -2.998  -2.998  -2.993  -2.897
 XI FOR STEP   20 TAU=TIME X KO=   .10000+00
   .2903   .3026   .3065   .3065   .3065   .3065   .3065   .3065
 SATURATION FOR STEP   20 TAU=  .100000+00
   .5890   .4720   .3031   .2894   .2915   .2941   .2972   .3022
 HYDRAULIC HEAD FOR STEP   20 TAU=  .100000+00
    .774    .052  -2.483  -2.990  -2.996  -2.989  -2.897  -3.000
 XI FOR STEP   30 TAU=TIME X KO=   .15000+00
   .2842   .2953   .3056   .3065   .3065   .3065   .3065   .3065
  SATURATION FOR STEP   30 TAU=  .150000+00
   .6187   .5546   .3845   .2930   .2917   .2941   .2972   .3022
  HYDRAULIC HEAD FOR STEP   30 TAU=  .150000+00
    .904    .528   -.883  -2.841  -2.993  -2.985  -2.897  -3.000
  XI FOR STEP   40 TAU=TIME X KO=   .20000+00
   .2800   .2900   .3005   .3064   .3065   .3065   .3065   .3065   .3065
```

-43-

```
SATURATION FOR STEP   40 TAU=  .20000D+00
  .6365   .5906   .5005   .3215   .2922   .2942   .2972   .3022   .3079
HYDRAULIC HEAD FOR STEP   40 TAU=  .20000D+00
  .974    .681    .158  -2.146  -2.971  -2.981  -2.897  -3.000  -3.000
XI FOR STEP  50 TAU=TIME X KO=  .2500D+00
  .2761   .2850   .2949   .3047   .3065   .3065   .3065   .3065   .3065   .3065
SATURATION FOR STEP   50 TAU=  .25000D+00
  .6511   .6153   .5576   .4246   .2977   .2943   .2972   .3022   .3079   .3128
HYDRAULIC HEAD FOR STEP   50 TAU=  .25000D+00
 1.031    .791    .441   -.534  -2.665  -2.975  -2.897  -3.000  -2.999  -3.000
XI FOR STEP  60 TAU=TIME X KO=  .3000D+00
  .2729   .2809   .2900   .2995   .3063   .3065   .3065   .3065   .3065   .3065
SATURATION FOR STEP   60 TAU=  .30000D+00
  .6605   .6326   .5905   .5128   .3431   .2949   .2972   .3022   .3079   .3128
HYDRAULIC HEAD FOR STEP   60 TAU=  .30000D+00
 1.066    .859    .580    .131  -1.753  -2.939  -2.895  -3.000  -2.998  -2.999
XI FOR STEP  70 TAU=TIME X KO=  .3500D+00
  .2699   .2770   .2854   .2948   .3041   .3065   .3065   .3065   .3065   .3065   .3065
SATURATION FOR STEP   70 TAU=  .35000D+00
  .6692   .6483   .6133   .5586   .4449   .3062   .2973   .3022   .3080   .3129   .3163
HYDRAULIC HEAD FOR STEP   70 TAU=  .35000D+00
 1.100    .920    .683    .345   -.497  -2.732  -2.891  -3.000  -2.998  -2.998  -3.000
XI FOR STEP  80 TAU=TIME X KO=  .4000D+00
  .2674   .2738   .2814   .2900   .2991   .3061   .3065   .3065   .3065   .3065   .3065
SATURATION FOR STEP   80 TAU=  .40000D+00
  .6765   .6578   .6304   .5903   .5132   .3598   .2983   .3022   .3080   .3130   .3163
HYDRAULIC HEAD FOR STEP   80 TAU=  .40000D+00
 1.128    .956    .750    .480    .061  -1.577  -2.834  -2.999  -2.997  -2.997  -2.999
XI FOR STEP  90 TAU=TIME X KO=  .4500D+00
  .2652   .2709   .2777   .2858   .2947   .3035   .3065   .3065   .3065   .3065   .3065   .3064
SATURATION FOR STEP   90 TAU=  .45000D+00
  .6831   .6664   .6460   .6119   .5594   .4556   .3120   .3024   .3080   .3130   .3164   .3196
HYDRAULIC HEAD FOR STEP   90 TAU=  .45000D+00
 1.153    .989    .811    .578    .248   -.484  -2.717  -2.995  -2.996  -2.995  -2.998  -3.000

XI FOR STEP 150 TAU=TIME X KO=  .7500D+00
  .2573   .2602   .2638   .2681   .2733   .2794   .2866   .2945   .3021   .3062   .3064   .3064
  .3064   .3064   .3064
SATURATION FOR STEP  150 TAU=  .75000D+00
  .6984   .6928   .6861   .6746   .6594   .6387   .6078   .5611   .4786   .3535   .3185   .3198
  .3232   .3269   .3310   .3354
HYDRAULIC HEAD FOR STEP  150 TAU=  .75000D+00
 1.211   1.090    .964    .820    .662    .483    .260   -.045   -.596  -2.080  -2.932  -2.995
 -2.996  -2.998  -3.000  -3.000
```

APPENDIX C

Extension of Solution Capabilities to Heterogeneous Soils

This appendix describes the changes which are necessary to the solution methodology used in the body of the report, in order to solve problems in which the saturated hydraulic conductivity of the soil varies in the vertical direction. A computer program has been written which do incorporate these changes. This program requires that the variation of the saturated hydraulic conductivity about some representative value be specified as input. The method used for handling problems with heterogeneous soils assumes that the same capillary pressure versus saturation and hydraulic conductivity relationships exist throughout the entire soil. In other words, the values for $Kr(p)$, $S(p)$, $\partial K_r/\partial p$ and $\partial S/\partial p$ do not vary with the soil heterogeneity. Rather the heterogeneity is described in terms of a variation of the saturated hydraulic conductivity. Granted, that this assumption cannot be used if different layers of soil are composed of vastly different original matter causing different layers to have vastly different hydraulic properties. For many situations in which the soil has been formed from basically the same mother material, however, this assumption is realistic. With this assumption the saturated hydraulic conductivity can be defined as

$$K_s = K_o K_v \tag{C-1}$$

in which K_o (a constant) is a representative value of the saturated hydraulic conductivity and K_v is a function of y. Darcy's Law can then be written as

$$v = - K_o K_v K_r \frac{\partial h}{\partial y} \tag{C-2}$$

Introduction of Eq. C-2 into the one-dimensional continuity equation gives the following equation in place of Eq. 1.

$$K_o \left[K_v \frac{\partial (K_r \frac{\partial h}{\partial y})}{\partial y} + K_r \frac{\partial h}{\partial y} \frac{\partial K_v}{\partial y} \right] = \eta \frac{\partial S}{\partial t} \tag{C-3}$$

After replacing h by the dependent variable ξ through the Kirchhoff transformation as before, leads to the following equation instead of Eq. 4 as the partial differential equation for which a solution is sought.

$$\frac{\partial^2 \xi}{\partial y^2} + \frac{1}{K_r} \frac{\partial K_r}{\partial p} \left[1 + \frac{1}{K_v} \frac{\partial K_v}{\partial y} \right] \frac{\partial \xi}{\partial y} + \frac{K_r}{K_v} \frac{\partial K_v}{\partial y} = \frac{\eta}{K_v K_r} \frac{\partial S}{\partial p} \frac{\partial \xi}{\partial \tau} \tag{C-4}$$

The finite difference operator for the Crank-Nicolson method from Eq. C-4 is

$$\xi_{i-1}^{j+1} - \left(\frac{2+\zeta}{1+\alpha} \right) \xi_i^{j+1} + \left(\frac{1-\alpha}{1+\alpha} \right) \xi_{i+1}^{j+1} = -\xi_{i-1}^{j} - \left(\frac{\zeta-2}{1+\alpha} \right) \xi_i^{j} - \left(\frac{1-\alpha}{1+\alpha} \right) \xi_{i+1}^{j} - 2 \frac{K_r}{K_v} \frac{\partial K_v}{\partial y} \tag{C-5}$$

in which

$$\zeta = \frac{2\eta \Delta y^2}{K_v K_r \Delta \tau} \frac{\partial S}{\partial p}$$

and

$$\alpha = \frac{\Delta y}{2K_r} \frac{\partial K_r}{\partial p} \left(1 + \frac{1}{K_r} \frac{\partial K_v}{\partial y} \right)$$

Note that Eq. C-5 is the same as Eq. 10 with two exceptions. First the final term in Eq. C-5 does not occur in Eq. 10, and second ζ and α are defined differently.

The boundary operators Eq. 12 through 14 will also be the same if the above definition for ζ and α are used with the exception of the operator (Eq. 12a) for the surface condition. In this last operator $K_v K_o$ replaces K_o in the second from the last term (See Eq. 12a).

With relatively minor changes from the procedure used to solve the homogeneous case, it is possible to obtain solutions to vertical infiltration problems into a soil whose hydraulic conductivity varies with depth.